**A25**  DEDUCE EXPRESSIONS TO CALCULATE THE NTH TERM OF LINEAR **AND QUADRATIC** SEQUENCES **(higher tier)**

**QUADRATIC SEQUENCES**

The *n*th term of a **quadratic sequence** has the form *an*2 + *bn + c* where *a*, *b* and *c* are numbers and *a* ≠ 0.

You need to be able to find the *n*th term of a quadratic sequence.

The **SECOND DIFFERENCES** of a quadratic sequence are **constant**.

The quadratic sequence with *n*th term *un* = *an*2 + *bn* + *c* has second differences equal to 2*a*.

Hence, if the second difference is constant then you are dealing with a quadratic sequence.

**EXAMPLE 1**

Show that the following is a quadratic sequence

8 13 20 29 40

Work out the first difference and then the second difference

8 13 20 29 40

+5

1st difference: Find the difference between each pair of terms

+2

+2

+2

+11

+9

+7

2nd difference: Use the 1st differences to find the 2nd differences

The second difference is constant, The 2nd difference is always 2

hence, it is a quadratic sequence.

**EXAMPLE 2**

Show that the following is **not** a quadratic sequence

1 5 15 31 53

Work out the first difference and then the second difference

1 5 15 31 52

+4

1st difference: Find the difference between each pair of terms

+6

+16

+10

+5

+4

+21

2nd difference: Use the 1st differences to find the 2nd differences

The second differences are **not** the same, The 2nd differences are 4, 6, 5

hence, it is **not** a quadratic sequence.

**EXERCISE1:**

Work out whether the following are quadratic sequences.

(a) 1, 5, 11, 19, 29, 41

(b) 2, 5, 8, 11, 14, 17

(c) 0, 8, 22, 41, 68, 98

(d) 2, 9, 20, 35, 54, 77

(e) 4, 1, 0, 1, 4, 9

(f) 6, 17, 36, 65, 98, 141

(g) 18, 37, 62, 93, 130, 173

(h) 3, 9, 23, 43, 75, 113

(i) −10, −4, 12, 38, 74, 120

(j) 17, 39, 69, 107, 153, 207

**Finding the *n*th term of a quadratic sequence**

To work out an expression for the *n*th term of a quadratic sequence we do the following:

1. Halve the second difference to get the value of *a* in *an*2

(e.g. if you had a second difference of +6 you would have +3*n*2).

2. Draw a table with 5 rows:

*n*, *n*2, *an*2, original sequence (*un*), *un* − *an*2 (known as **the residue**)

3. The residue will either be constant or a linear sequence.

If it is a linear sequence then work out its formula.

5. Finally add the formula for the residue to *an*2.

**EXAMPLE 3**

Find the *n*th term of the sequence 1 3 6 10 15

Work out the first difference and then the second difference

1 3 6 10 15

+2

+1

+1

+1

+5

+4

+3

The second difference is constant so it is a **quadratic** sequence of the type

*un* = *an*2 + *bn* + *c.*

The coefficient of *n*2 in the *n*th term is **half the second difference**.

The second differences are +1, so *a* = 0.5

Once you have worked out the value of *a*, draw a table like this one.

This final row is known as the **residue**

Original sequence

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *n*  *a* = 0.5 | 1 | 2 | 3 | 4 | 5 |
| *n*2 | 1 | 4 | 9 | 16 | 25 |
| 0.5*n*2 | 0.5 | 2 | 4.5 | 8 | 12.5 |
| *un* | 1 | 3 | 6 | 10 | 15 |
| *un* – 0.5*n*2 | 0.5 | 1 | 1.5 | 2 | 2.5 |

The residue forms an **arithmetic sequence**.

The arithmetic sequence 0.5, 1, 1.5, 2, 2.5… has *n*th term *un* = 0.5*n*.

This is the last part of the *n*th term of your quadratic sequence.

Hence, *n*th term = 0.5*n*2 + 0.5*n*

**EXAMPLE 4**

Find the *n*th term of the sequence 3 13 27 45 67

Work out the first difference and then the second difference

3 13 27 45 67

+14

+18

+22

+10

+4

+4

+4

The second difference is constant so it is a **quadratic** sequence of the type

*un* = *an*2 + *bn* + *c.*

The coefficient of *n*2 in the *n*th term is **half the second difference**.

The second differences are +4, so *a* = 2

Once you have worked out the value of *a*, draw a table like this one.

Original sequence

Residue

*a* = 2

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *n* | 1 | 2 | 3 | 4 | 5 |
| *n*2 | 1 | 4 | 9 | 16 | 25 |
| 2*n*2 | 2 | 8 | 18 | 32 | 50 |
| *un* | 3 | 13 | 27 | 45 | 67 |
| *un* – 2*n*2 | 1 | 5 | 9 | 13 | 17 |

The final row forms an **arithmetic sequence**.

Goes up in 4s so 4*n*

Term before 1 is

1 − 4 = 3

The *n*th term of the arithmetic sequence 1, 5, 9, 13, 17 Goes up in 4s so 4*n*

is 4*n* − 3

*un* = 4*n* − 3

This is the last part of the *n*th term of your quadratic sequence.

Hence, *n*th term = 2*n*2 + 4*n* − 3

**EXAMPLE 5**

Find the *n*th term of the sequence 7 16 31 52 79

Work out the first difference and then the second difference

7 16 31 52 79

+15

+21

+27

+9

+6

+6

+6

The second difference is constant so it is a **quadratic** sequence of the type

*un* = *an*2 + *bn* + *c.*

The coefficient of *n*2 in the *n*th term is **half the second difference**.

The second differences are + 6, so *a* = 3

Once you have worked out the value of *a*, draw a table like this one.

Residue

Original sequence

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *n* | 1 | 2 | 3 | 4 | 5 |
| *n*2  *a* = 3 | 1 | 4 | 9 | 16 | 25 |
| 3*n*2 | 3 | 12 | 27 | 48 | 75 |
| *un* | 7 | 16 | 31 | 52 | 79 |
| *un* – 3*n*2 | 4 | 4 | 4 | 4 | 4 |

The final row just has + 4 in each column.

So this is the last part of the *n*th term of your quadratic sequence.

Hence, *n*th term = 3*n*2 + 4

**EXERCISE 2:**

1. The first six terms of a sequence are 6, 9, 14, 21, 30, 41

(a) What are the differences between consecutive terms?

(b) What are the differences between the differences (the second row of differences)?

(c) What does your answer to part (b)tell you?

(d) Find the general term of the sequence.

2. Here are the first 6 terms of a sequence.

2 9 20 35 54 77

(a) Calculate the second differences for this sequence.

(b) Find a simpler sequence by subtracting 2*n*2 from each term

(c) Find an expression, in terms of *n*, for the *n*th term of the simpler sequence.

(d) Find an expression, in terms of *n*, for the *n*th term of the original sequence.

3. Here are the first 6 terms of a sequence.

6 9 14 21 30 41

(a) Calculate the second differences for this sequence.

(b) Find an expression, in terms of *n*, for the *n*th term of this sequence.

4. (a) Work out the second differences of the following sequence.

6 17 36 63 98

(b) Hence, or otherwise, find an expression, in terms of *n*, for the *n*th term of this sequence.

5. Here are the first 6 terms of a sequence.

9 4 -5 −18 −35

(a) Calculate the second differences for this sequence.

(b) Find an expression, in terms of *n*, for the *n*th term of this sequence.

(c) Hence show that the 25th term of the sequence is −1215

6. Show that 862 is the 20th term of the quadratic sequence

7 16 29 46 67

(Hint: Find an expression for the *n*th term and then substitute for *n* = 20)

7. Here are the first 5 terms of a quadratic sequence

3 9 17 27 39

Jean says that 161 is a term of this sequence.

(a) Is Jean correct? Give a reason for your answer.

Nav says that all the terms in the sequence are odd numbers.

(b) Is Nav correct? Give a reason for your answer.

**EXERCISE 3:**

1. Find an expression, in terms of *n*, for the *n*th term of each of these quadratic sequences

(a) 2 3 6 11 18 (b) 3 10 19 30 43

(c) 5 11 19 29 41 (d) –1 0 3 8 15

(e) –2 4 12 22 34 (f) 3 9 17 27 53

(g) –2 4 14 28 46 (h) 1 5 15 31 53

(i) 3 6 11 18 27 (j) −19 −15 −9 −1 9

(k) 4 10 20 34 52 (l) 2 9 22 41 66

(m) 2 12 26 44 66 (n) 14 67 122 179 238

(o) 9 4 -5 −18 −35 (p) 6 10 12 12 10

2. Here are the first 5 terms of a quadratic sequence.

5 14 27 44 65

Find an expression, in terms of *n*, for the *n*th term of this quadratic sequence.

3. Here are the first 5 terms of a quadratic sequence.

4 11 22 37 56

Find an expression, in terms of *n*, for the *n*th term of this quadratic sequence.

**EXERCISE 4:**

1. Here are some patterns made from square slabs.

(a) Write down an expression, in terms of *n*, for the *n*th term of this sequence.

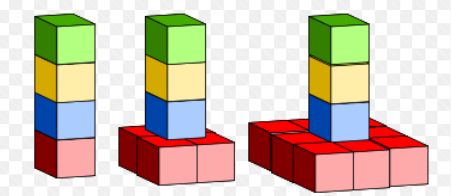
(b) Jane says that 75 is a term in the quadratic sequence.

Is Jane correct? Give a reason for your answer.

2. Here are some patterns made from tiles.

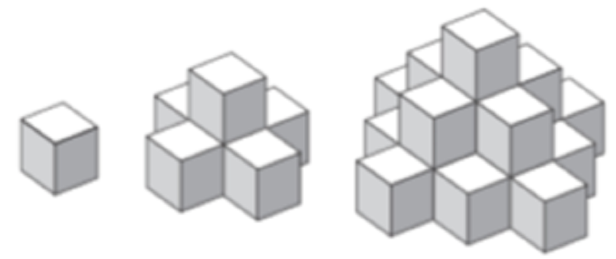
Find an expression, in terms of *n*, for the *n*th term of this sequence.

3. Here are some patterns made from cubes.



Find an expression, in terms of *n*, for the *n*th term of this sequence.

4. Here are some patterns made from cubes.



(a) Work out the number of cubes needed to make the next pattern.

(b) Find an expression, in terms of *n*, for the *n*th term of this sequence.

5. Here is a sequence of patterns made from centimetre squares.

Find the number of centimetre squares in the 100th pattern.

6. Here is a sequence of patterns made from centimetre squares.

Find an expression in terms of *n* for the number of centimetre squares in the *n*th pattern.

**ANSWERS**

**Exercise 1**

(a) yes (b) no (c) no

(d) yes (e) no (f) no

(g) yes (h) yes (i) yes

(j) yes

**Exercise 2**

1. (a) 3, 5, 7, 9, 11 (b) 2

(c) quadratic sequence (d) *n*th term = *n*2 + 5

2. (a) 4 (b) 0, 1, 2, 3, 4

(c) *n* – 1 (d) *n*th term = 2*n*2 + *n* – 1

3. (a) 2 (b) *n*th term = *n*2 – 2

4. (a) 8 (b) *n*th term = 4*n*2 − *n* + 3

5. (a) −4 (b) *n*th term = −2*n*2 + *n* + 10

(c) 25th term = −2(25)2 + 25 + 10 = −1215

6. *n*th term = 2*n*2 + 3*n* + 2

20th term = 2(20)2 + 3(20) + 2 = 862

7. *n*th term = *n*2 + 3*n* – 1

3, 19, 17, 27, 39, 53, 69, 87, 107, 129, 153, 179

(a) Jean is incorrect as 162 does not lie in the sequence.

(b) Nav is correct as all the numbers are odd.

**Exercise 3**

1. (a) *n*2 – 2*n* + 3 (b) *n*2 + 4*n* – 2

(c) *n*2 + 3*n* + 1 (d) *n*2 – 2*n*

(e) *n*2 + 3*n* – 6 (f) *n*2 + 3*n* – 1

(g) 2*n*2 – 4 (h) 3*n*2 – 5*n* + 3

(i) *n*2 + 2 (j) *n*2 + *n* – 21

(k) 2*n*2 + 2 (l) 3*n*2 – 2*n* + 1

(m) 2*n*2 + 4*n* – 4 (n) *n*2 + 10*n* + 3

(o) –2*n*2 + *n* + 10 (p) –*n*2 + 7*n*

2. 2*n*2 + 3*n*

3. 2*n*2 + *n* + 1

**Exercise 4**

1. (a) 2*n*2 (b) No, as *n* is not an integer

2. *n*2 + 3

3. *n*2 + 3

4. (a) 40 (b) 4*n*2 – 7n + 4

5. *n*th term = *n*2 + *n*

100th term = 5050

6. 2*n*2– 2*n* + 1