Getting Students to Create Boundary Examples

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Students spend most of their time *assenting* to what other people assert mathematically. This does not imply that they have blindly accepted it; indeed they may have worked hard in order to agree with it. But agreement is not the same as understanding. If they are to make mathematical sense themselves, then they need to be able to *assert* things for themselves. They need to use technical terms with facility to express their ideas. In particular, when students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions.

Stimulated by some ideas of Zygfryd Dryszlag (1984), we have outlined a wide range of questions and prompts which mathematicians use in their thinking, and which could be used by teachers to invite students to think mathematically (Watson & Mason, 1998). In particular, we are interested in the effect on students' understanding of asking and expecting them to generate their own examples.

Boundary Examples

One class of these prompts is the construction of boundary examples. Askew & Wiliam (1995) refer to *only just* and *very nearly* examples: an example is *just* an example if any change in the example causes it to become a non-example, and a *nearly* example needs but one further adjustment in order to become an example. Although initially attractive as an idea, it is often a moot point as to when a non-example can be tweaked or wrenched into becoming an example and vice versa.

We suggest that a mathematical concept does not always have clear boundaries of meaning which ought, conventionally, to be elaborated in definitions by certain conditions and statements of properties. For instance, it is well known that students tend to retain their intuitions about continuity, limits such as $.\dot{9}$, elements of sets such as groups as objects not operations, etc., despite being introduced to, and even appearing to handle competently, formal definitions. Indeed Fischbein (1987) believed that intuition is never displaced, merely overlaid. Even if one does not entirely agree with him, the fact remains that students' sense of, or concept image for a technical term is often an interwoven mixture of details from formal definitions and intuitions. Nevertheless, the idea that there are 'edges' to mathematical ideas which are worth exploring is a powerful one.

We prefer to use the notion of *boundary (classes of) example.* Boundary examples distinguish between having and not having a specified property. If students are only offered well-behaved examples, or examples which have additional, but irrelevant, features, then the reason for careful statements of conditions to a theorem or definition might pass them by and they may well develop the idea that it is possible to have ambiguous or undecided cases. If, for instance, students have only experienced monotonically increasing or decreasing sequences, they may logically believe that no sequence can attain its limit. If they have also encountered sequences such as {1, 1, 1, 1, …}, and appreciated the generic nature of such an example, they may be more aware of the infinitely many ways a sequence might incorporate a constant subsequence, or might take its 'limiting value' in the sequence. With such experience, students are less likely to focus on the irrelevant feature of

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increase or decrease. Note that we are not talking about specially constructed pathological examples here, but a class of rather ordinary examples which may extend the boundaries of the students' understanding. How can students be helped to appreciate the importance of such examples? How can they be helped to focus attention on relevant features?

We offer the following conjecture:

> If you cannot construct boundary examples for a theorem or a technique, then you do not fully appreciate or understand it.

We are not claiming that construction of examples constitutes understanding, just that it makes a very useful contribution to achieving familiarity and competence. Students can be prompted to make deeper sense of theorems and techniques if they themselves have constructed examples and non-examples.

Constructing examples which meet constraints

A common approach to counter-examples is to provide students with examples, viz. :

- Draw a sketch of a continuous real-valued function with domain [0, 1] and co-domain [0,1] which is not differentiable at a point in the interior of that interval.
- Write down a polynomial which passes through the points (1, 1), (2, 3), and (3, 5).
- Write down an infinite sequence which diverges even more slowly than $\{1/n\}$.

These are standard objects which are offered to students in courses. The trouble is that they are usually provided by the teacher not the students, so many students do not spend time working on the example to see why it has the claimed properties, nor for what conjectures, and why, it provides a counter-example. Each is both an opportunity to get the students to construct examples themselves, and an opportunity to generalise:

- Which subsets of the reals can be the set of points at which a function is not differentiable? Build up a class of such functions which show the possibilities.
- Find a general method for writing down the most general polynomials which pass through a finite set of specified points, such as (1, 2), (3, 5), (5, 8), (8, 13), … . Build up a general procedure for answering such questions.
- Build up a collection of different ways to modify a sequence so that it converges or diverges more quickly (more slowly) than a given sequence.

Here there are two important features: students constructing their own examples, so that they have to check conditions and at the same time develop fluency, and students contemplating the general class of such examples. The next section offers a particularly good way to promote this.

Boundary Examples

A powerful version of example construction arises when constraints can be meaningfully compounded, so that examples are required which meet all but one of the constraints. We find it sufficient to accumulate the constraints in order, and then to work backwards to find boundary examples. Viz. :

- Draw a sketch of a real-valued function with domain [0, 1], and co-domain [0,1];
- Draw a sketch of a real-valued function with domain [0, 1], and co-domain [0,1] which is also continuous;
- Draw a sketch of a real-valued function with domain [0, 1], and co-domain [0,1] which is also continuous and which also has a maximum on [0, 1] at one end-point;
- Draw a sketch of a real-valued function with domain [0, 1], and co-domain [0,1] which is also continuous;, which has a maximum on [0, 1] at one end-point and which has a minimum on [0, 1] at one end-point;
- Draw a sketch of a real-valued function with domain [0, 1], and co-domain [0,1] which is also continuous, which has a maximum on [0, 1] at one end-point, which has a minimum on [0, 1] at one end-point, and which has a local maximum and a local minimum in the interior.

Now comes the interesting part. Work your way back *up* the list, making sure that at each stage your example *does not* satisfy the next (more demanding) condition. Thus your first example must be a discontinuous function, your last but one will have the extrema at the endpoints but will not have local maxima or minima in the interior.

Comment

One of the effects of an exercise like this is that students become aware that they (and their teachers) often use an overly particular 'generic example' when they hear some term used. ‘Example’ can be interpreted to mean ‘a special example’ or ‘an obvious example’. Indeed, one variant is to ask students at each stage to construct a particular example, then a peculiar example (eg. one which no-one else in the class is likely to think of), and then (if appropriate) a general, or at least maximally general example. The effect of seeking a ‘peculiar’ example is to open up for consideration the range of possibilities, the scope of generality (Bills 1996).

By constructing a boundary example students are forced to extend their example-space in order to complete the task. So one effect is that students become more aware of the range of possibilities from which they are choosing when they select an example, and this is a precursor to expressing generality.

Other Examples

- Write down a group of order 8; which is also Abelian; which is the direct product of two non-trivial subgroups; which has an element of order 4. Working backwards up the list, the first challenge is to find a group of order 8 which is a direct product but with no element of order 4, while the last is to write down a group not of order 8.
- Write down a sequence of positive reals; which converges to 2; which takes on the value 2 infinitely often; for which infinitely many values are not 2; for which for any positive integer n there is a term beyond the n^{th} which is greater than $1/n$.
- Write down a linear transformation $R^3 \to R^3$; which is singular; which has (0, 0, 1) in the kernel; which also sends (1,0,0) –> (1, 0,0); which also has (0,1,0) has an eigenvector; with eigenvalue 3.

Again most of the force of the work comes when working back up the list seeking boundary examples.

Comments

Of course it would be possible to discuss and describe the most general object possible at each stage. It would also be desirable sometimes to introduce constraints which are impossible to fulfil. In the case of the function sketching, discussion of exactly what ‘local maximum’ means is likely to occur (do you accept a function which is constant at the maximum value near the endpoint as having a local maximum?). Provoking students to discuss extreme or special cases such as empty intersections and unions, circles of zero or infinite radius, triangles with an angle of 0, etc. contributes to their sense of the range of possibilities covered by the conditions.

Final Comments

Using a sequence of increasing constraints has the added virtue that it displays mathematical objects as the result of *freedom* (to choose any example in a wide range of possibilities) within *constraints.* For example, solving linear equations is really seeking the points which started as being general or free, but then have to satisfy a number of constraints; finding a method for making a geometrical construction can similarly be seen as starting with initial freedom or generality, and then adding constraints.

We have found that many students do not appreciate the range or scope of choice of objects which are permitted by a theorem. Most theorems can be seen as a description of something which is *invariant-amidst-change*, and the theorem states the scope and range of change that are permitted. But if students have not tried to construct examples for themselves, have not probed the role of various conditions in making a theorem or technique work, then they are unlikely to use it appropriately, and probably unlikely to think of using it at all!

Finally, there is the ubiquitous concern about time. The first time you ask students to construct an example for themselves, most will probably not succeed, at least without a fair amount of time. How then can the material be ‘covered’? Our answer is that over a period of time working on getting students to construct examples, they will become much better at it, and in the process, much more proficient in the mathematics. Consequently they will begin to learn more efficiently, so that the pace of the course can accelerate (but neither constantly positive nor uniformly!). In effect, we cannot afford *not* to invest the time needed in order to enable students to appreciate the ideas to which they are being exposed.

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