Editorial Manager(tm) for Educational Studies in Mathematics
Manuscript Draft

Manuscript Number: EDUC668R1

Title: Shedding Light On and With Example Spaces

Article Type: Special Issue: Examples

Section/Category:

Keywords: example; exemplification; example spaces

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Manuscript Region of Origin:

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Response to Reviewers: The paper has been re-organised so as to reduce repetition and so as to clarify points raised by reviewers.

Shedding Light On and With Example Spaces

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1. Abstract

Building on the papers in this special issue as well as on our own experience and research, we try to shed light on the construct of *example spaces* and on how it can inform research and practice in the teaching and learning of mathematical concepts. Consistent with our way of working, we delay definition until after appropriate reader experience has been brought to the surface and several 'examples' have been discussed. Of special interest is the notion of *accessibility* of examples: an individual's access to example spaces depends on conditions and is a valuable window on a deep, personal, situated structure. Through the notions of *dimensions of possible variation* and *range of permissible change*, we consider ways in which examples exemplify and how attention needs to be directed so as to emphasise examplehood (generality) rather than particularity of mathematical objects. The paper ends with some remarks about example spaces in mathematics education itself.

Keywords: examples, exemplification, example spaces

2. Examples

The widespread use of 'examples' in mathematics textbooks from the earliest recorded time is a manifestation of the common insight that it is through the appreciation of familiar examples that abstractions become reified (Sfard 1994). Just as with natural language, meaning arises mainly from encountering instances in use, while definitions provide a reference against which to test those uses. Put another way, definitions function as generalisations or abstractions whose meaning emerges through experience of particular familiar instances and from which learners re-construct the generalisation/abstraction for themselves. Lacock & Inglis (this volume) demonstrate different ways that some learners use examples: empirically, to test out conjectures, and imagistically, to illustrate, clarify or ensure access to meaning or method.

Examples can therefore usefully be seen as cultural mediating tools between learners and mathematical concepts, theorems, and techniques. They are a major means for 'making contact' with abstract ideas and a major means of mathematical communication, whether 'with oneself', or with others. Examples can also provide context, while the variation in examples can help learners distinguish essential from incidental features and, if well selected, the range over which that variation is permitted.

Mathematical objects only become examples when they are perceived as 'examples *of* something': conjectures and concepts, application of techniques or methods, and higher order constructs such as types of proof, use of diagrams, particular notation or other support, and so on. The fundamental construct is the act of *seeing something as an example of* *some 'thing'*. Thus the number 36 can be seen as an instance of use of place-value, as an even number, as a number divisible by 3, as a square number, as a triangular number, and so on. It only exemplifies through perception. This is the fundamental ontological move required of learners, producing the experience of 'reification' (Sfard 1991).

In mathematics, Rissland (Michener 1978) distinguished between start-up examples (motivation for and initiation into a topic), reference examples (used to inform and develop intuition), model or generic examples, and counterexamples (to slight alterations in theorems). Zazkis & Leikin (this volume) added pertinent non-examples of concepts. In each case their role is to provide something specific and familiar on which to base and explore new ideas and to check out the role of constraints or conditions with respect to a definition. By working on and through sets of 'examples', learners encounter nuances of meaning, variation in parameters and other aspects that can change.

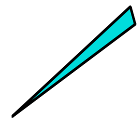
2.1 Examples, Counterexamples & Non-examples

In a mathematical context there is little difference between an example and a counterexample: it all depends where your attention is anchored, and what you are attending to. Thus an example of a concept or a theorem is a counterexample to an inappropriate variation of the concept definition or the theorem; a counterexample to a variation in a definition or theorem illustrates its role and importance but can also provide an example for a revised definition or assertion. For instance, $|x|$ is an example of a continuous function on the reals, but a counterexample to the conjecture that all continuous functions are also everywhere differentiable and a motivation for Weierstrasse's construction of a continuous but nowhere differentiable function (Hardy 1916). Similarly 0.9 is a non-example of a number whose square is larger than it, an example of a number which is positive but whose square is not greater than 1, or a counterexample to the conjecture that "squaring makes larger". The fact that -1 and 2 have a difference of 3 is a counterexample to the conjecture that $-\underline{a}$ and $\underline{a}$ have the same remainder on dividing by 3, but an example of how to extend the definition of remainders to negative numbers consistently. What matters most is that learners are aware of what features of an object make it an example, and what features can be varied to form a class of similar or related examples. The examplehood of an object thus resides in the experience of the learner and the typicality that it evokes in terms of what can change and what must remain (relatively) invariant.

Bruner and colleagues (Bruner *el al* 1956) were among many researchers who tried to find clear and unambiguous links between the use of examples and non-examples for concept acquisition, without much success. Hershkowitz (1989) explored the same issue in a geometrical context with mixed results, while Wilson (1986) and Petty & Janssson (1987) explored optimal sequencing of examples and non-examples. Tirosh & Levenson (this volume) address the question of whether non-examples are associated with prototypes, or whether they have a looser structure based on tinkering with examples in relation to properties, and provide a more extensive account of various attempts to address this issue in the literature.

2.2 Representativeness

How representative is a single mathematical object as an example of some 'thing'? Modern mathematics is founded on the notion of equivalence relations. Any member of a class of equivalent items serves equally well to 'represent' the class to which it belongs. This explains the role played by so-called generic examples in mathematics (Courant 1991). However outside of mathematics, there is greater variety. A large log serves as a chair at a campfire, but not often in a sitting room; the word *furniture* is more likely to conjure up *chair* than *stool*. Lakoff (1987) drew on Rosch's analysis of the role of prototypes in categorisation (Rosch 1973, 1977) to contrast equivalence categories with radial categories in which prototypes are considered to have various degrees of centrality. Radial categories may be useful for appreciating what learners go through in the process of making sense of examples they are given in mathematics. Whereas for an experienced mathematician, specific mathematical objects may be fully representative of a large class of related objects through various construction techniques, for learners some examples are more representative, more perspicuous, than others.

Fischbein (1987) used the term *figural* to refer to the dual nature of geometrical concepts that are understood both through perception of examples, and through logical or ideal features, giving rise to the notion of a *figural concept*. However, when perception dominates logic, there is potential misunderstanding: learners may include inappropriate perceptual features in their extension of the concept beyond the examples. Thus, many text books used to draw all their squares with edges parallel to the edges of the page, misleading learners into thinking that a tilted square is not a square at all, but a 'diamond'. Few textbooks illustrate the notion of a triangle with a figure like  which some students see not as a triangle but as a stick (Sfard 2001), and some still show all their triangles with one edge parallel to the bottom of the page, leading to restricted notions not only of triangle, but of altitude as necessarily vertical.

Fischbein's notion can usefully be extended to any situation (diagrammatic or symbolic) in which some perceptual features may not be relevant, and may intrude or even dominate. A person's conception can be judged *figural* if, having been illustrated by examples with irrelevant attributes, learners implicitly or explicitly associate those attributes with the concept. Put another way, in a *figural conception*, some dimensions of possible variation and/or some ranges of permissible change have been inappropriately included or excluded.

Herein lies the danger of prototypes, which can be compounded when examples are constructed on the spur of the moment, because chosen parameters may not only be ambiguous, but in conflict, turning the 'example' into a non-existent inconsistent object (see Zodik & Zaslavsky, this volume). Thus, two important features of example spaces worth probing are which attributes a learner is aware can be varied, and what range of that variation the learner recognizes as appropriate.

Dienes (1963) claimed that learners need at least three examples in order to get a sense of a concept. Variation theory (see section 3.1) suggests that where learners depend on examples in order to locate dimensions of possible variation, those examples must be chosen very carefully so as to prompt learners to discern which aspects can vary and which are structural.

> The fundamental problem is whether generalization should take place simultaneously on a broad front, or on several narrow fronts followed by abstraction into a broader front later. Younger children seem to find generalizations on a narrow front, that is within certain well-defined fields, easier. (Dienes 1963 p. 160)

Freudenthal (1983 p. 33) disagrees with Dienes' concentration on physical embodiments of mathematical structure as a precursor to internalisation. He prefers treating mathematical objects as mental objects upon which one acts, rather than slavishly following a sequence of *physically enactive* and *mentally iconic* in order to reach the *abstractly symbolic* as implied by Bruner's (1966) three modes of representation. Familiarity with examples develops through mental action on both particular and general objects.

3. Example Spaces

Examples of mathematical objects come as classes, though some classes may have only one or two members. For example, asked to think of an irrational number, most learners think first of √2 or π. If there is no further discussion, these may constitute their accessed example space (accessed at that time, even though other examples may be accessible at other times). However, learners are usually aware that they can change various parameters to achieve other examples. In the case of irrationals, learners may know that irrationality is preserved by multiplying by any rational (other than 0), and adding any rational. They may also be aware that they can change the 2 in √2, though they may not be articulate about the case of 4, 9, 16, and so on, until their attention is drawn specifically to them as counterexamples to universality of the reasoning, or as non-examples of the reasoning (Mason, Drury & Bills 2007, see also Movshovitz-Hadar & Hadass 1990). Few may have thought about changing the square root to other roots.

Alcock & Inglis (this volume) refer in passing to how example spaces may consist of diagrams as well as other kinds of objects. Thus, mention of trigonometric ratios may trigger specific geometric examples (isosceles, half an equilateral, or a more generic right-angled triangle in a specific orientation).

When the notion of example space is extended into meta-mathematics, proofs can be seen as examples of the format or structure of mathematical reasoning; specific formats of tasks can be seen as examples of ways of working on mathematics (Watson & Shipman this volume); definitions can be seen as examples of the process of defining (Zazkis & Leikin, this volume); and research papers in mathematics education can be seen as exemplifying different approaches to scholarly exploration of issues in effective teaching and learning.

3.1 Dimensions of Possible Variation and Range of Permissible Change

It seems that a 'thing' is perceived as an 'example' in two similar but subtly different ways. The first is when one is aware of possible and permitted variations that maintain examplehood. Sometimes referred to as an instance of the mathematical theme of 'invariance in the midst of change' (Mason & Johnston-Wilder 2006, Mason 2008), this 'awareness' may be anywhere on a spectrum from largely implicit to articulately explicit. Thus, looking at one triangle, it is possible to have a sense of how the vertices can be varied without affecting the sum of the exterior angles, or the sum of the interior angles. The second way of perceiving something as an example is when the specific is experienced as a particular case of a generality. Thus a square can be perceived of as a special case of a rectangle, without immediate experience of a rectangle varying; a particular quadratic can be experienced as an instance of a quadratic, again without any direct experience of varying coefficients. Of course full appreciation of generality also requires awareness of permissible change.

Watson & Mason (2005) use the term *dimensions of possible variation* to refer to the features of an example that learners recognize as eligible for change, without losing examplehood. This awareness may not be immediate, but may be triggered by teacher probes. Associated with each such aspect or dimension is the *range of permissible change.* Thus with √2 as an example of an irrational come the dimensions of scaling and translating by a rational (the range of permissible change in both dimensions being all rationals apart from multiplication by zero,) as well as the number inside the square root (certainly any non-square positive integer, but potentially a much greater range of permissible change) and the exponent (which could be any non-integral rational but potentially could have an even greater range).

The adjective *possible* was introduced as an addition to Marton's construct of *dimensions of variation* (see Marton & Booth 1997) as a reminder that different people at different times may be aware of different dimensions that can be varied. The construct *range of permissible change* was added because people often have an unnecessarily restricted sense of the scope of change available in any given dimension. Certainly teachers are usually aware of more dimensions and of a greater range of permissible change within each dimension than come to mind for learners, at least in the early stages of making sense of a concept.

Watson & Shipman (this volume) present clear evidence of learners unfamiliar with surds having access to a space of tractable examples involving small positive integers inside and outside the square-root sign. In an unfamiliar situation, they made use of familiar actions (grid multiplication) to try to see what might be going on structurally as they struggled to achieve a rational product. Watson & Shipman also show how perception of invariance in the midst of change can be directed through gesture, symbols, layout, diagram, dynamics, colour and font, in the examples of 'substitution' or 'symbol interpretation'.

3.2 Accessible Example Spaces

Some vast collection of examples may be in the back of someone's mind, but at any moment only a limited number of examples is accessed. Furthermore, those may be at the front of the mind, or they may be just below the surface and require some effort to reconstruct in detail. Sometimes there is a general 'sense' of examples without specific details, while at other times a few clear examples come to mind immediately, setting off a trail of metonymic triggers and metaphoric resonances associated with concepts, techniques, or other features, or with ways to vary or tinker with those examples.

Accessibility can be illustrated by two related tasks:

> Please construct a number which, when you subtract 1, leaves a number divisible by 7.
>
> Please construct a number that is 1 more than a multiple of 7.

The two tasks often trigger quite different experiences. The second task is usually seen as an explicit construction recipe and hence experienced as straightforward, whereas the first is a property to be satisfied and is sometimes experienced as sufficiently difficult to require trial and error. Furthermore, some people see the 1 and the 7 as *dimensions of possible variation* while others do not (Mason *et al* 2004). Pedagogically, the first can be seen as a representative of an even wider class of tasks intended to provoke expression of generality and use of that generality to justify a conjecture:

> Construct a number which, when you subtract 1 is divisible by 2; and when you subtract 1 from the quotient the result is divisible by 3; and when you subtract 1 from the new quotient the result is divisible by 4. Why must your number be divisible by 3?

Watson & Mason (2005) suggested a pantry metaphor for ways in which examples are stored and accessed: some are at the front and used frequently, while others are farther back, awaiting an opportunity to be used. Some may have been acquired when an interesting object was encountered, even without a specific concept or technique to illustrate, while others may be acquired when first meeting a new term, theorem or technique.

Storage is one thing; access is another. It turns out that different examples are available at different times according to what triggers them: what appears at the front of the pantry shelves depends upon the situation and recent use. It is helpful therefore to use the term *accessible example space* as a reminder that what comes to mind in the moment is *situated*: it is dependent on many factors including the context, the cue or trigger, and the state of the individual. As mentioned earlier, when asked to think of an irrational number, learners may first have π or √2 come to mind, but these are, themselves, cues, and can change the learner's state, perhaps giving access to further transcendental and algebraic examples. In a group, one person's example can trigger access to a further class of examples for someone else. Furthermore, each time a connection is made it is strengthened and more likely to come to mind in the future. Tsamir *et al* (this volume) show something of the variety of ways in which an example space may be organised, with implications for what is then most, and what least, accessible.

3.4 Example Construction and Example Space Enrichment

Examples are linked not only by association, such as belonging to one or more classes of related examples, or illustrating or exemplifying the same or similar topic or construct, but also by being generated using construction methods that go beyond dimensions of possible variation. Thus important aspects or components of example spaces are the construction methods that can be used to form classes, and modifications of classes, according to need. Watson & Mason (2005) present a number of different task-structures that involve learners constructing examples.

Watson & Shipman (this volume) connect this process with what Simon and colleagues describe as reflection on the relationship between activity and its perceived effect (Simon *et al* 2005, see also Tzur in press). When you gain control over a range of examples by seeing them as instances of a generality, or as components with which to generate further examples, your relationship with the topic changes. The pantry metaphor is particularly apt as a reminder that example construction is often a matter of bricolage, of tinkering with familiar examples in order to create an example meeting specific constraints.

3.5 Probing Example Spaces

Aspects of example spaces can be revealed through use of probes involving learners constructing examples (Watson & Mason 2002, 2005). To show how examples might give clues about a person's thinking, Zazkis asked a participant in a discussion group to construct many examples of irrational numbers between 100 and 200. He first scaled $\sqrt{2}$, then added integers to it, and only then started to use π and other numbers. His trajectory illustrates how examples are not a simple memory lookup. In this case, relying on memory alone would have turned up a blank stare since canonical examples of irrational numbers are rarely between 100 and 200. The combinatorial approach, starting with some known irrationals and applying various transformations in combination, provides access to unlimited numbers of examples. The eventual switch to totally new starting places (e.g., $\sqrt{10001}$, which *starts* within the required interval) suggests that he has indirect access to many other examples through a combination of familiar examples and ways to construct new ones.

Rowland (this volume) adds to the evidence provided by Zazkis & Leikin (this volume) that example construction, whether as an explicit task, or when done by a teacher in the heat of the moment, reveals a good deal about the person's accessible example space in that situation, and hence the scope of their awareness and the focus of their attention. It may therefore be useful to distinguish a special class of *accessible instructional example spaces*.

However, from the learner's perspective there is really only one experienced space, which may be triggered differently by exploration (communication with oneself) and by need to communicate with others.

Although Abdul-Rahman (2005) and Zazkis & Leikin (2007), as well as Zazkis & Leiken (this volume) and Alcock & Inglis (this volume) show how example spaces can be used as a research tool for revealing understanding, care must be taken. The adage *absence of evidence is not evidence of absence* applies particularly when asking learners to reveal their grasp of a concept: the fact that they don't display an example does not imply that it is not within their accessible space, just that they have not perceived a reason to express it.

3.6 Example Spaces Summary

In summary therefore, an *example space* is an experience of having come to mind one or more classes of mathematical objects together with construction methods and associations. There may be internal structure in the form of connections between objects or between classes, and there may be associative links with concepts, theorems, procedures, and so on. Each accessed space is generated by varying aspects which the person is aware of as variable, over ranges which the person sees as permissible, and by tinkering with structural aspects to make new compound collage-like objects.

At any moment what comes to mind is only part of a more subterranean space below the surface of consciousness. The most important point is that example spaces are not static, but rather dynamic and evolving. What comes to mind can trigger access to further examples and construction techniques according to the richness of connections, and can lead to an even richer experience in the future.

The notion of a *space of examples* is not new. Michener (1978) used the term *example space* to refer to a canonical or universal space in the world of mathematics. Zaslavsky & Peled (1999) used the term to refer to the collection of mathematical examples to which someone has access, in order to explain teachers' difficulties in generating counterexamples. Watson & Mason (2002, 2005) stressed the situatedness of examples that come to mind. They promoted the use of the term *example spaces* and developed techniques for prompting learners to become aware of, and to enrich, the spaces to which they might have access in the future (see also Dahlberg & Housman, 1997). They also distinguished several kinds of examples, including worked-examples (extensively studied, by, among others, Atkinson *et al*. 2000, Leinhardt 2001, Renkl *et al*. 1998, and Renkl 2002). Yet more kinds are distinguished in Zazkis & Leikin (this volume).

The richer the connections and the construction methods, and the more often the 'pantry' is worked through, the more meaningful will be the experienced sense of the concepts, theorems, techniques etc..

4. Role in Informing Teaching and Learning

Example spaces as described here are inescapable components of the experience of learners. As such, learning more about a topic includes gaining access to further examples or constructions for such examples, as well as enriching the interconnections and extending the triggers and resonances affording access to those spaces. Teaching effectively includes making use of tasks and interactions through which learners gain access to examples, to construction methods, and of course to mathematically relevant features of different examples.

Simply 'giving' examples and construction techniques is rarely sufficient for most learners. Most learners need to (re)construct examples in order to populate their example space, and benefit from explicit stimuli to reflect on the parameters of that space: the dimensions of possible variation and associated ranges of permissible change. Marton's theory of *variation* both explains how learners can learn concepts from examples and how teachers can structure examples for maximum effect (Watson & Mason 2006). People are naturally attuned to detecting variation in objects in close temporal and physical proximity. Variation of appropriate complexity (not too many factors at once, nor too few) in sufficiently close temporal and physical proximity so that learners experience them as juxtaposed, prompts learners' natural sense-making powers to generalise. This generalising comes about through appreciating dimensions of (possible) variation and ranges of permissible change, by seeing generality through the particular.

Inviting learners to construct examples subject to constraints, whether aesthetic ones such as 'simple', complicated, extreme or general, or mathematical ones that exploit features of the concept, theorem or technique, reveals some of the structural complexity of learners' example spaces. When learners compare their examples, they often extend and enrich their example space to take account of other people's dimensions of possible variation or range of permissible change.

Sometimes teachers offer an object to illustrate a point. Such an illustration may not be intended to be prototypical, paradigmatic, generic or representative, but may instead merely be illustrative of some specific property or relationship (Sowder 1980). Yet learners may not appreciate this subtle difference between an illustration of a specific property and an example of a broader concept, and may take the example to be paradigmatic. Most important however is that access to what is being illustrated be enhanced by analysing, or prompting learners to analyse, what features can vary and in what ways. It is tempting to think that an example 'speaks for itself', but novices usually need to be clear about what is being exemplified and what is not; what is structural and general, and what is particular.

4.1 Pedagogical Effectiveness through Perspicuity

As Schwarz & Hershkowitz (1999), Tsamir, Tirosh, & Levenson (this volume), and Rowland (this volume) demonstrate so clearly, there can be great differences in the pedagogical effectiveness of examples according to the perspicuity and clarity. For example,

> In a lesson, a teacher developed the equation of a circle with radius 2, centred at (a, b). However, the number 4 in the equation $(x - a)^2 + (y - b)^2 = 4$ could be interpreted as double the radius or as the square of the radius.

The choice of 2 for the radius adds unnecessary ambiguity and the possibility of inappropriate conjectures as to the structural significance of the 4. A similar problem arises when a circle of radius 2 is used for applying the formulae for circumference and area: some instances of 2 are parameters, and some are structural, and it is important to know which is which. The fact that 2 + 2 = 2 x 2 renders 2 a poor choice of parameter in most situations.

Both Zaslavsky and Rowland (this volume) contribute to the catalogue of ways in which examples, especially those chosen on the spur of the moment in a lesson, can be particularly unhelpful to learners. Rowland's example of a novice teacher choosing 4 – 2 = 2 rather than, say, 5 – 2 = 3 where the numbers are all different, or using (1, 1) when working on the meaning of coordinates, highlights the importance of carefully chosen examples. A similar confusion is amplified if the circle equation is developed using particular coordinates for the centre, especially ones such as $a = 1$, $b = -1$ because of unintended ways to associate the constants in the equation with the coordinates of the centre based on superficial syntactic relationships. Of course, one might deliberately use such an example in assessment, to probe learners' understanding by tempting them to make inappropriate associations.

Rowland (this volume) points out that novice teachers may be dominated by considerations of affect (motivation, engagement, interest) when seeking variety in examples to offer learners but that this may interfere with, or be in conflict with, making careful use of mathematically structured variation in order to promote and prompt generalisation and abstraction.

The notion of an *example space* can serve as a reminder that choices can be made when something comes to mind as an instantiation, particularity, or example of 'some thing'. Thus when preparing for a lesson or in the midst of a lesson, it is useful for a teacher to be aware of the space of examples from which to choose rather than using the first that comes to mind. Similarly, when studying, working on tasks or exploring mathematical consequences, it is useful to become aware of a class of examples from which to choose rather than being restricted to the first thing that comes to mind.

The notions of *dimension of possible variation* and *range of permissible change* are valuable to teachers as reminders that it is important to make it explicitly clear to learners what features of an object make it an example: that is, what features are structural, and what features can be altered in what ways. As Rowland (this volume) observes, when examples of a procedure, technique or method are displayed (usually through worked examples), learners are invited to generalise; when examples are used to illustrate concepts, learners are invited to abstract. In the first case they are extending and enriching their example space of potential problem types through general classes of similar problems; in the second they are enriching and extending their example space to provide representative models of properties that have been abstracted and axiomatised or defined. Rowland also notes that the use of variation in structured exercises varies considerably from country to country and text to text.

4.2 Conditions in which Example Spaces Flourish and Develop

Example spaces flourish best within a classroom rubric (Floyd *et al* 1981), what Cobb *et al* (1996) refer to as *socio-mathematical norms*, compatible with learner construction and interaction. Such a classroom atmosphere promotes conjecturing (Mason *et al* 1982; see also LeGrand 1993) rather than focusing on a few paradigmatic or generic examples. Examples are analysed for features that contribute to appreciating examplehood, construction and tinkering, and ways to generate further classes. Learners who know that everything said is said as a conjecture that may require modification develop a propensity not only to look for both examples and counterexamples, but to use these to modify conjectures. Learners who are aware of looking for what is the same and what is different as a learning and problem solving strategy (Brown & Coles 2000, Marton 2006) develop rich and extended example spaces, while those who try to master the examples they are given as templates are condemned to a restricted and potentially misleading example space.

Watson & Shipman (this volume) show how within an appropriate classroom rubric it is possible to use exploratory construction tasks to provoke learners into encountering key aspects of a concept or topic. Example spaces can grow through exploration as well as through direct construction tasks.

5 Example Spaces in Mathematics Education as a Domain of Enquiry

Researchers and teacher educators depend on examples to enrich their understanding of pedagogic constructs. Definitions in mathematics education are particularly elusive because there is no axiomatic foundation on which to base new constructs. Unfortunately, mathematics educators have consistently failed to develop any taken-as-shared practice for negotiating the relevance and exemplariness of examples of the constructs that they use. A classic instance is the way in which an example, proffered by Vygotsky in connection with his emerging notion of a *zone of proximal development*, became an unfortunate if not significantly misleading example (Valsiner 1988). According to Valsiner, whereas Vygotsky intended the *ZPD* to refer to the potential for development from being able to carry out actions under guidance to being able to initiate those actions independently, it has, through the commonly quoted extract, been widely used to refer to changes in behaviour directed by a more experienced other (a projection into a *zone of proximal behaviour*), without reference to conscious control (Mason *et al* 2007).

One important role that the construct of *example spaces* could play in the refinement of mathematics education as a discipline is as a reminder that what determines the use of a concept is the example space one associates with it. The *science of education* (Gattegno 1970) and the *discipline of noticing* (Mason 2002) were both articulated as contributions to the more careful exchange and negotiation of examples or instances of pedagogic constructs, in order to turn mathematics education from a confusion of multiple uses of similar terms and multiple terms with similar meanings, into a productive and developing discipline.

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Figure of Triangle

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