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***Micro-essays, inspired by the "Goldenberg"-Mason paper***

## Examples and language

Examples are examples *of* something. When we present examples of a *thing*—triangles, irrational numbers, Banach spaces—it is clear we are talking about a *class* or *category*. It may be less obvious (but is still true) that even when we present examples of a *process*—for example, a particular subtraction algorithm or a method for taking a derivative—the example represents a class of behaviors, not all identical, but with common features.[1]

A major *purpose* of examples is to help communication. In fact, it is not clear that there is any *other* purpose. Even in high-level formal mathematical presentations, a definition is often followed by some examples, often to clarify terms or implications of the formal specifications in the definition.

Because *category* is such a central idea—examples are examples *of* something—and because communication is the primary or sole purpose of examples, we cannot talk sensibly about the nature and purpose of examples in mathematics learning without connecting it to ideas whose origin rests primarily in research in linguistics and cognition.

## Definitions vs. examples

Examples are important in mathematical communication partly because definitions, by themselves, are generally incapable of conveying a concept "from scratch". This is universally true outside of mathematics, where category boundaries are often fuzzy, but is true within mathematics as well, especially for learners,[2] whose limited experience means that more ideas feel to them to be arising "from scratch".

In everyday conversation, definitions are of little real help. Try to think, for example, how you would define "chair" to include all the different kinds of objects, wooden, plastic, stuffed, formal, etc., that are "chairs," and how to exclude superficially similar

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[1] In mathematics, moreover, the distinction between process and object is fuzzier than elsewhere. In one context, a function may adequately be thought of as a set of steps for processing an input and generating an output. For beginners, that may be the only context they recognize, and thus, the entire meaning of *function*. Slightly more generally (but still process-like), a function is a rule that maps elements of one set to elements of another. More atomic than a "set of steps," this image is of a single object—a machine, perhaps—but it still *acts* on other objects, elements of two sets, connecting them in some way. But at a yet more abstract level, we can see functions apart from whatever sets they might relate: in this image, the functions, themselves, are set elements, manipulable by higher-order functions that can, themselves, be elements of a set, and so on.

[2] In a way, this qualification is both non-specific and unnecessary. When those with great experience encounter a definition for the first time, they, too are "learners". But it is also often the case that they encounter new definitions within a context in which they are already facile, and have at least some examples already. Typically, also, mathematical definitions are "simple" in that they present a new term (category) as a combination of two or more features (other categories) already both totally unambiguous and for which the reader can generate numerous examples, or simply not understand the definition! It is that latter fact—that one cannot use a definition unless one can already bring to mind examples of all its defining terms—that makes definitions, even in mathematics, even for experts, inadequate if one assumes one is starting "from scratch". Mathematics requires definitions, but experts can use them only because they are *not* starting from scratch.

objects that are not chairs. Or think how to define "cat." Alternatively, imagine that you did *not* know these words; then look in a dictionary to see how much you must already know in order to understand the definition! Finally, think how little that definition really contains of the "cat" in your head. Definitions are not easy routes to meaning, even for adults, until one already has a fair idea what the word means from use in context—that is, from examples! It's not uncommon for adults to notice, when asked (perhaps by a child) the meaning of a word that they've long understood and used, that they don't really know, and have to look it up. For casual use, context and experience are enough to give us "the general idea" of a word, and make it *useful* even if we cannot give a definition.

This is as true for children as it is for adults. Young children acquire vocabulary at an astonishing rate, a full 50% of their expected adult vocabulary by the age of five! They do that entirely from use in context, generally without anything resembling didactic instruction. New words are acquired extremely rarely from a definition and never solely from one. And, in fact, when children do look up definitions before they have the general idea—a kind of vocabulary-learning that occurs only in schools—the results are generally strange. They look up "extinguish", see that it means "put out", and write "before I go to bed each night, I extinguish the cat".[3]

Examples might be thought of as bits of context—ways to give information other than "saying what the word means"—allowing vocabulary-learning in school to grow at least slightly closer to the natural language learning at which children are so adept. Examples allow teachers to *use* a word communicatively until students are able to use it as well. Teachers can *use* the word rather than explaining it because the example provides the context and carries the meaning. Only then, when the students already have a *rough* meaning from communicative use in context can one effectively clarify the meaning formally with other words, through discussion and/or definition.

### Casual vs. mathematical communication

There are important differences between mathematical communication and casual communication. We can put up with some ambiguity in most common communication, which is fortunate because ambiguity is almost inevitable, even in law, which tolerates ambiguity poorly. But no ambiguity at all[4] is tolerable in formal communication of mathematical results, reasoning, and other ideas among mathematicians.[5]

In mathematics, definitions are essential, because examples, alone, never nail down meanings precisely enough for the careful use mathematics makes of words. Even so—

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[3] Adults *can* get some words from definitions, but even they mostly rely on meanings from context and usage, which is why a good glossary for teachers should probably give correct and incorrect usage, where appropriate, along with definitions.

[4] This is not to say that all mathematical terms have only one meaning. Terms *out of context*, can be ambiguous, but mathematical communication supplies a context. Within that context, formal mathematical communication requires complete lack of ambiguity.

[5] Communication of such ideas *in teaching* is a different matter. Because learners' knowledge is more limited, any context, itself a category, is likely to remain incomplete and its boundaries fuzzy. The context, then, supplies less constraint on the meaning of new terms. Such communication necessarily remains partially ambiguous.

and especially in elementary school—definitions really don't quite "work" until one mostly understands anyway. When a word is already roughly understood—like the fully familiar word that you suddenly realize you don't *really* know the meaning of—only then can a definition help clean up the details, refining and making precise what one already knows in a fuzzy and approximate way. And in elementary school, even then it is hard to get a definition precise enough to do the job without using words and ideas that the child doesn't know!

> A fifth grade teacher asked her student teacher to review the ideas of factors and multiples with the class. The student teacher started by asking the children to list the factors of 25. At first, the students confused the idea, and listed 50, 75, 100... She clarified that she was looking for "numbers that could be multiplied to make 25". She was surprised when the children listed only 5, and then stopped. It is not uncommon for children to "miss the obvious" because it doesn't seem to be "interesting" enough! There doesn't seem to be enough "multiplying" going on in $1 \times 25$. But she asked for more and, after a long pause, one child said, tentatively, "two times twelve-and-a-half?" after which another child practically exploded with the answer "four times six-and-a-quarter!" The class was proudly showing their knowledge of fractions, which they'd just studied.
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> So, how many factors *does* 25 have? The student teacher was then unsure, herself. Just the number 5? Or 1, 5, and 25? Or are the zillions of other possibilities legitimate, too, because they are "numbers that could be multiplied to make 25"? It all depends on what we mean by factor. For that reason, definitions in mathematics are essential.[6]

Mathematics builds new ideas on already established ideas. We can't build a new idea on "it depends what you mean", so we need, right at the start, to *agree* on what we mean. Moreover, we can't share our discoveries with others unless they agree on the *same* meanings for the words we use. We can't, for example, claim that "a prime number is a number that has only two factors, 1 and itself" unless we agree that even though, $2 \times 12\frac{1}{2}$ is 25, twelve-and-a-half is not a "factor" of 25.

### Mason's paradox in exemplification

This is quite like the observation that definitions don't help until we already have the general idea. "Examples" remain discrete one-of-a-kind events until we see features that unite them. Thus—whether or not the right number of examples is three, as Dienes posited—examples don't exist *at all* in the singular; they are not "examples" of anything unless they come in sufficient numbers to allow the learner to induce common features, or they follow some other information, like a definition, that calls attention to those features. In other words, we cannot see them as examples until they are examples *of* something we already have (at least vaguely) in mind; their examplehood depends on our already perceiving the class that they exemplify.

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[6] And also, for that reason, definitions for teachers *must* include the borderline cases, the non-cases, and the common confusions. It is not sufficient to say "One of two or more expressions that are multiplied together to get a product" as math.com does. Even entries that say "whole number that exactly divides…" don't quite make clear what children will ask: isn't 25 divided by 2 *exactly* 12½?

**The context of an example**

Asking a student to *circle all the parallelograms* in a collection of figures that includes non-parallelograms, prototype parallelograms, and various special-case parallelograms that are often thought *not* to be parallelograms because they have their own special names, we take each special-case figure that the student does not circle as evidence that the student's understanding of *parallelogram* is incomplete.

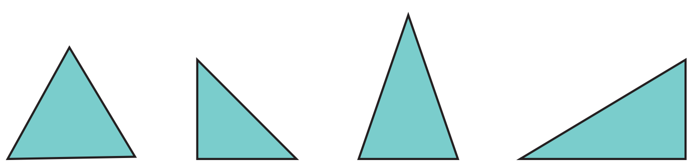
But if we ask a student to *draw* a parallelogram, we expect *not* to see the special cases that we'd hope the student would circle in the previous example. And, in fact, if we *do* get them as responses, we might well take that as evidence of error or incomplete understanding.

The context—*circle* examples or *give* an example—determines how we interpret the results.

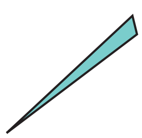
If one asks a student to give an example of a parallelogram and the student responds by drawing a square, one cannot tell whether the student is exceptionally clever and is trying to show a grasp of the hierarchy that gives this very familiar object—one of the first shapes children learn to name—a new family name, or whether the student simply doesn't know what a parallelogram is, except that it is a shape, and so is picking the first shape that came to mind as being at least partially responsive.[7] In other words, we can't tell whether the student's response is overly specific—a special case of a parallelogram—or overly general—a shape, indicating only the knowledge that that is the general category in which *parallelogram* is a special case.

**Examples, extreme examples, non-examples, and the inadequacy of all that!**

A person whose "concept image" (Vinner, 19??) of a triangle is built from examples like:



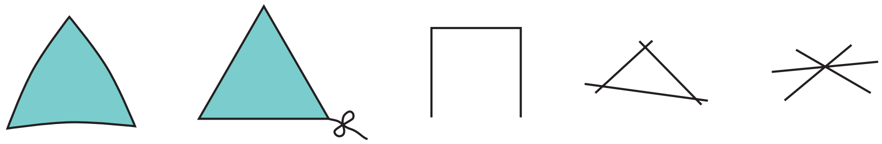
is not likely to recognize



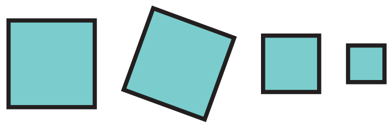
as a triangle, so that "extreme" example must be given, too. Research shows that, indeed, many children do not recognize the extreme case.

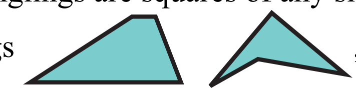
It is important, also, to give non-triangle examples. Three of these are often selected as triangles by beginners, as they are quite close to the prototype examples. The upside-down box and the asterisk are rarely selected, but do conform to common incomplete "definitions" of triangles as "shapes made with three lines."

[7] That is, drawing a square instead of saying "seven" or "Peru" tells us that the student *does* know something important about parallelograms, even if the student has little more to say about them.

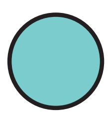


But after a rough idea is given by examples, extreme examples, and non-examples, it is still the case that definition is needed. It is easiest to show why by example, using a whimsical category called “smanglings.” Imagine developing this category inductively, through examples.

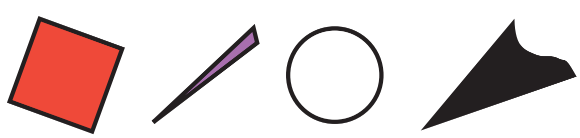
Each of these is a smangling: 

So far, it would seem that smanglings are squares of any size or orientation. When we learn that these, too, are smanglings , we realize that the category is broader, and might decide that smanglings are probably quadrilaterals of any kind. But these are also smanglings!  So, *now* what do smanglings appear to be?

For pedagogical purposes—maybe for any purposes other than sheer perversity—the order in which the examples were presented was terrible. In order for examples not to be misleading (as we deliberately were here!), it is important that they be as varied as possible—varying each of the dimensions of possible variation through a reasonable sampling of the range of permissible change. It is *often* (but not always) the case that the variation should come *early*, so that the first impressions are not misleading, as they were above.[8]

We cheated. This, too, is a smangling:  *Now* what do smanglings appear to be?

It is tempting, now, to conclude that smanglings can be any closed two dimensional shape. But it is *not* logical to draw that conclusion, just as we could not (logically) conclude, before this example, that smanglings could be any polygon. The right answer is “We can’t tell, because we haven’t seen anything yet that *isn’t* a smangling.”

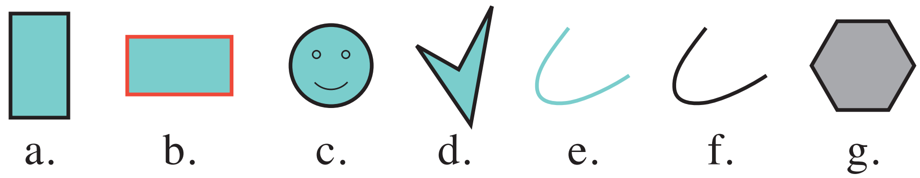
So, here are a few non-smanglings: .

Without these examples, it is doubtful that we would have guessed that colour mattered, and it would also have been illogical to make such a guess, as there was no evidence one

[8] But it is easy to find cases when one does want to limit the initial complexity. For example, it *may* be that we want students’ first impressions of functions to be limited to the “school” kind—single-variable, real-number-in-real-number-out, definable with polynomials. It *may* be that we want to enlarge that idea sooner than we do, but facing the full zoo of monsters all at once may not be educationally sound.

way or the other. Here is another non-smangling: . Apparently the border must be visible!

At this point we do have a lot of information. Let's see how far it gets us. Which of these is a smangling?




We can be fairly sure that **a** and **d** are smanglings, and we can be equally sure that **g** is not.[9] But what about the rest? Case **c** is the right colour and has a black border, but we don't know whether decorations on the inside are allowed. No non-example rules them out, but no *example* shows them. This exercise (because it is purely inductive, without definitions) is more like science than mathematics: we can hypothesize pending further testing, but we cannot decide. It seems likely that case **e** should be ruled out as it has no black border, but case **f** is harder to rule out. It has the black line, and though it is not the right colour on the inside, it *can't* be because it has no inside! We have no evidence about this case. It certainly doesn't fit the examples, but it isn't ruled out by the non-examples, either. We have the same problem with case **b**. It is the right colour on the inside, and the border is visible, but we've learned that colour matters. Maybe the colour of the border matters, too, but we simply don't know.

Without belaboring it, here's the point: just as definitions without examples are generally insufficient to convey meaning, so are examples without definitions. No matter how numerous and varied our examples and non-examples are, unless they are exhaustive (i.e., the set of smanglings is finite, and we have encountered every one of them as either an example or non-example), examples alone are insufficient to allow us to decide all cases, because they provide no way of knowing whether or not some perverse exception lurks among the cases that have not been seen. But the examples—and especially the task of trying to choose among the unknowns and then defend that choice—make it much easier to *perceive* the dimensions of possible variation and the range of permissible change. One advantage for students of encountering this meta-mathematical idea is that it helps motivate what otherwise often seems like bizarre over-particularity in the wording of definitions. There is a lot we must say to define a smangling in a way that allows us to decide, definitively and without question, which of the unknowns is and isn't a smangling.

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[9] Of course, even of this we cannot be certain, as there might be more than one acceptable colour, and gray might be one of the accepted ones. There might also be some perverse exception that rules out non-square rectangles. But it would, indeed, be perverse to have such arbitrariness, especially without evidence.