**A7** WHERE APPROPRIATE, INTERPRET SIMPLE EXPRESSIONS AS FUNCTIONS WITH INPUTS AND OUTPUTS**; INTERPRET THE REVERSE PROCESS AS THE ‘INVERSE FUNCTION’; INTERPRET THE SUCCESSION OF TWO FUNCTIONS AS A ‘COMPOSITE FUNCTION’ (THE USE OF FORMAL FUNCTION NOTATION IS EXPECTED) (higher tier)**

In GCSE Mathematics equations are written as shown below:

 *y* = 3*x* + 4 *y* = *x*² + 5

Sometimes a different notation is used which is called **function notation**.

We often use the letters f and g and we write the above equations as

 f(*x*) = 3*x* + 4 g(*x*) = *x*² + 5

**EXAMPLE 1**

Using the equation *y* = 3*x* + 4, find the value of *y* if

(a) *x* = 4 (b) *x* = *−*6

(a) *y* = 3(4) + 4 = 12 + 4 = 16 Substitute for *x* = 4 in the equation

 (b) *y* = 3(–6) + 4 = –18 + 4 = –14 Substitute for *x* = −6 in the equation

**EXAMPLE 2**

f is a function such that f(*x*) = 3*x* + 4

Find the values of

(a) f(4) (b) f(*−*6)

(a) f(4) = 3(4) + 4 = 12 + 4 = 16 Substitute for *x* = 4 in the equation

(b) f(*−*6) = 3(–6) + 4 = –18 + 4 = –14 Substitute for *x* = −6 in the equation

**EXAMPLE 3**

g is a function such that g(*x*) = 2*x*² – 5

Find the values of (a) g(3) (b) g(*−*4)

(a) g(3) = 2(3)² – 5 = 18 – 5 = 13 Substitute for *x* = 3 in the equation

(b) g(–4) = 2(–4)² – 5 = 32 – 5 = 17  Substitute for *x* = −4 in the equation

**EXAMPLE 4**

The functions f and g are defined for all real values of *x* and are such that

 f(*x*) = *x*² – 4 and g(*x*) = 4*x* + 1

Find (a) f(−3) (b) g(0.3)

(c) Find the two values of *x* for which f(*x*) = g(*x*).

(a) f(−3) = (−3)² – 4 = 9 – 4 = 5 Substitute for *x* = −3 in the equation f(*x*)

(b) g(0.3) = 4(0.3) + 1 = 1.2 + 1 = 2.2 Substitute for *x* = 0.3 in the equation g(*x*)

(c) *x*² – 4 = 4*x* + 1 Put f(*x*) = g(*x*)

 *x*² – 4 – 4*x* – 1 = 0 Rearrange the equation as a quadratic = 0

 *x*² – 4*x* – 5 = 0 Simplify

 (*x* – 5)(*x* + 1) = 0 Solve the quadratic by factorising

 *x* – 5 = 0 or *x* + 1 = 0

 *x* = 5 or *x* = –1

**EXERCISE 1:**

1. The function f is such that f(*x*) = 5*x* + 2

Find (a) f(3) (b) f(7) (c) f(−4)

 (d) f(−2) (e) f(−0.5) (f) f(0.3)

2. The function f is such that f(*x*) = *x2* − 4

Find (a) f(4) (b) f(6) (c) f(−2)

 (d) f(−6) (e) f(−0.2) (f) f(0.9)

3. The function g is such that g(*x*) = *x*3 − 3*x*2 *–* 2*x +* 1

Find (a) g(0) (b) g(1) (c) g(2)

 (d) g(−1) (e) g(−0.4) (f) g(1.5)

4. The function f is such that 

Find (a) f(0) (b) f(1) (c) f(2)

 (d) f(−1) (e) f(−0.7) (f) f(1.5)

5. f(*x*) = 3*x*2 – 2*x* – 8

Express f(*x* + 2) in the form *ax*2 + *bx*

6. The functions f and g are such that

 f(*x*) = 3*x* – 5 and g(*x*) = 4*x* + 1

(a) Find (i) f(−1) (ii) g(2)

(b) Find the two values of *x* for which f(*x*) = g(*x*).

7. The functions f and g are such that

 f(*x*) = 2*x*2 – 1 and g(*x*) = 5*x* + 2

(a) Find f(−3) and g(−5)

(b) Find the two values of *x* for which f(*x*) = g(*x*).

**COMPOSITE FUNCTIONS**

A **composite function** is a function consisting of 2 or more functions.

The term composition is used when one operation is performed after another operation.

For instance:

+ 3

× 5

*x*

*x* + 3

5(*x* + 3)

This function can be written as f(*x*) = 5(*x* + 3)

Suppose f(*x*) = *x*2 and g(*x*) = 2 + 3*x*

**What is fg(*x*)?** Nowfg(*x*) = f[g(*x*)]

This means apply g first and then apply f.

 fg(*x*) = f(2 + 3*x*) = (2 + 3*x*)²

It is then substituted into f(*x*)

g(*x*) is replaced with 2 + 3*x*

**What is gf(*x*)?**

This means apply f first and then apply g.

 gf(*x*) = f(*x*²) = 2 + 3*x*²

It is then substituted into g(*x*)

f(*x*) is replaced with *x*2

**NOTE:** The composite function **gf(*x*)** means apply f first followed by g.

**NOTE:** The composite function **fg(*x*)** means apply g first followed by f.

**NOTE:**  fg(*x*) can be written as fg and gf(*x*) can be written as gf; fg is not the same as gf.

**EXAMPLE 5**

f and g are functions such that f(*x*) =  and g(*x*) = 3 – 2*x*

Find the composite functions (a) fg (b) gf

(a) fg = fg(*x*) = f (3 − 2*x*) Do g first: Put (3 − 2*x*) instead of g(*x*)

 =  Substitute (3 − 2*x*) for *x* in 

(b) gf = gf(*x*) = g Do f first: Put  instead of f(*x*)

 = 3 – 2 = 3 –  Substitute  for *x* in (3 − 2*x*)

**EXAMPLE 6**

f(*x*) = 7 – 2*x* g(*x*) = 4*x* – 1 h(*x*) = 3(*x* – 1)

Find the following composite functions: (a) gf (b) gg (c) fgh

(a) gf = gf(*x*) = g(7 − 2*x*) Do f first: Put (7 − 2*x*) instead of f(*x*)

 = 4(7 − 2*x*) − 1 Substitute (7 − 2*x*) for *x* in 4*x* − 1

 = 27 − 8*x* Simplify 28 − 8*x* − 1

(b) gg = gg(*x*) = g(4*x* − 1) Put (4*x* − 1) instead of g(*x*)

 = 4(4*x* − 1) − 1 Substitute (4*x* − 1) for *x* in 4*x* − 1

 = 16*x* − 5Simplify 16*x* − 4 − 1

 (c) fgh = fgh(*x*) = fg[3(*x* − 1)] Put 3(*x* − 1) instead of h(*x*)

 = fg (3*x* − 3) Expand 3(*x* − 1)

 = f [4(3*x* − 3) − 1] Substitute (3*x* − 3) for *x* in 4*x* − 1

 = f (12*x* − 13)Simplify 12*x* − 12 − 1

 = 7 − 2(12*x* − 13) Substitute (12*x* − 13) for *x* in 7 − 2*x*

 = 33 − 24*x* Simplify 7 − 24*x* + 26

**EXAMPLE 7**

f(*x*) = 7 – 2*x* g(*x*) = 4*x* – 1 h(*x*) = 3(*x* – 1)

Evaluate (a) fg (5) (b) ff (−2) (c) ghf (3)

(a) fg(5) = f(20 − 1) = f(19) Substitute for *x* = 5 in 4*x* − 1

 = 7 − 2(19) = −31 Substitute for *x* = 19 in 7 − 2*x*

(b) ff(−2) = f[7 − 2(−2)] = f(11) Substitute for *x* = −2 in 7 − 2*x* and simplify

 = 7 − 2(11) = −15 Substitute for *x* = 11 in 7 − 2*x* and simplify

(c) ghf(3) = gh(7 − 6) = gh(1) Substitute for *x* = 3 in 7 − 2*x* and simplify

 = g[3(1 − 1)] = g(0) Substitute for *x* = 1 in 3(*x* − 1) and simplify

 = 4(0) − 1 = −1 Substitute for *x* = 0 in 4*x* − 1 and simplify

**EXAMPLE 8**

f(*x*) = 3*x* + 2 and g(*x*) = 7 − *x*

Solve the equation gf(*x*) = 2*x*

 gf(*x*) = g(3*x* + 2) Put (3*x* + 2) instead of f(*x*)

 = 7 − (3*x* + 2) g's rule is subtract from 7

 = 5 − 3*x* Simplify 7 − 3*x* − 2

 5 − 3*x* = 2*x* Put gf(*x*) = 2*x* and solve

 5 = 5*x* Add 3*x* to both sides

 *x* = 1

**EXAMPLE 9**

 **(more challenging question)**

Functions f, g and h are such that

f: *x* 4*x* − 1 g: *x* , *x* ≠ −2 h: *x* (2 – *x*)2

Find (a)(i) fg(*x*) (ii) hh(*x*) (b) Show that fgh(*x*) = 

(a) fg(*x*) = f

Substitute for g(*x*)

 = 4 – 1

f’s operation is × 4 − 1

 = 

Simplify using a common denominator of *x* + 2 denominator o

 = 

(b) hh(*x*) = h(2 – *x*)2  Substitute for h(*x*)

h’s operation is subtract from 2 and then square

 = [2 – (2 – *x*)2]2

 = [2 – (4 – 2*x* + *x*2)]2  (2 − *x*)2 = (2 − *x*)(2 − *x*) = 4 − 2*x* + *x*2

 = (−2 + 4*x* − 2*x*2 )2

Put (4*x* – 1) for f(*x*)

(c) hgf(*x*) = hg(4*x* – 1)

 = h()

Substitute (4*x* – 1) for *x* in g(*x*)

 = 

4*x* – 1 + 2 = 4*x* + 1, so put 4*x* + 1 for *x* in h(*x*)

 =  Simplify using a common denominator of 4*x* + 1

 = 

 = 

**EXERCISE 2:**

1. Find an expression for fg(*x*) for each of these functions:

(a) f(*x*) = *x* − 1 and g(*x*) = 5 – 2*x*

(b) f(*x*) = 2*x* + 1 and g(*x*) = 4*x* + 3

(c) f(*x*) =  and g(*x*) = 2*x* − 1

(d) f(*x*) = 2*x*2  and g(*x*) = *x* + 3

2. Find an expression for gf(*x*) for each of these functions:

(a) f(*x*) = *x* − 1 and g(*x*) = 5 – 2*x*

(b) f(*x*) = 2*x* + 1 and g(*x*) = 4*x* + 3

(c) f(*x*) =  and g(*x*) = 2*x* − 1

(d) f(*x*) = 2*x*2  and g(*x*) = *x* + 3

3. The function f is such that f(*x*) = 2*x* − 3

Find (i) ff(2) (ii) Solve the equation ff(*a*) = *a*

4. Functions f and g are such that

 f(*x*) = *x*2 and g(*x*) = 5 + *x*

Find (a)(i) fg(*x*) (ii) gf(*x*)

(b) Show that there is a single value of *x* for which fg(*x*) = gf(*x*) and find this value of *x*.

5. Given that f(*x*) = 3*x* – 1, g(*x*) = *x*2 + 4 and fg(*x*) = gf(*x*), show that *x*2 – *x* – 1 = 0

6. The function f is defined by f(*x*) =, *x* ≠ 0

 Solve ff(*x*) = −2

7. The function g is such that g(*x*) =  for *x* ≠ 1

(a) Prove that gg(*x*) = 

(b) Find ggg(3)

8. Functions f, g and h are such that f(*x*) = 3 – *x*, g(*x*) = *x*2 − 14 and h(*x*) = *x* − 2

 Given that f(*x*) = gfh(*x*), find the values of *x.*

**INVERSE FUNCTIONS**

The function f(*x*) = 3*x* +5 can be thought of as a sequence of operations as shown below

× 3

+ 5

*x*

3*x*

3*x* +5

Now reversing the operations

÷ 3

− 5

*x*

*x −* 5



The new function, , is known as the **inverse** function.

Inverse functions are denoted as f −1(*x*).

**EXAMPLE 10**

Find the inverse function of f(*x*) = 3*x* − 4

 *y* = 3*x* − 4 **Step 1:**  Write out the function as *y* = ...

 *x*= 3*y* − 4 **Step 2:**  Swap the *x* and *y*

 *x* + 4 = 3*y* **Step 3:** Make *y* the subject

  = *y*

 f −1(*x*) =  **Step 4:** Instead of *y* = write f −1(*x*) =

**EXAMPLE 11**

Find the inverse function of f(*x*) = 

 *y* =  **Step 1:**  Write out the function as *y* = ...

 *x*=  **Step 2:**  Swap the *x* and *y*

7*x* = *y* − 2 **Step 3:** Make *y* the subject

 7*x* + 2 = *y*

 f −1(*x*) = 7*x* + 2 **Step 4:** Instead of *y* = write f −1(*x*) =

**EXAMPLE 12**

Find the inverse function of f(*x*) = 

 *y* =  **Step 1:**  Write out the function as *y* = ...

 *x*=  **Step 2:**  Swap the *x* and *y*

 *x*2 = *y* + 4 **Step 3:** Make *y* the subject

 *x*2 − 4 = *y*

 f −1(*x*) = *x*2 − 4 **Step 4:** Instead of *y*= write f −1(*x*) =

 **RULES FOR FINDING THE INVERSE f −1(*x*):**

 **Step 1:**  Write out the function as *y* = ...

 **Step 2:**  Swap the *x* and *y*

 **Step 3:** Make *y* the subject

 **Step 4:** Instead of *y*= write f −1(*x*) =

**EXERCISE 3:**

1. Find the inverse function, f −1(*x*), of the following functions:

(a) f(*x*) = 3*x* − 1 (b) f(*x*) = 2*x* + 3

(c) f(*x*) = 1 – 2*x* (d) f(*x*) = *x*2 + 5

(e) f(*x*) = 6(4*x* – 1) (f) f(*x*) = 4 − *x*

(g) f(*x*) = 3*x*2 − 2 (h) f(*x*) = 2(1 − *x*)

(i) f(*x*) =  (j) f(*x*) = 

2. The function f is such that f(*x*) = 7*x* − 3

(a) Find f −1(*x*).

(b) Solve the equation f −1(*x*) = f(*x*).

3. The function f is such that f(*x*) = 

(a) Find f −1(*x*).

(b) Solve the equation f −1(*x*) = f(*x*).

4. The function f is such that f(*x*) = ,  *x* ≠ −4.

 Evaluate f −1(*x*). [Hint: First find f −1(*x*) and then substitute for *x* = −3]

5. f(*x*) = , *x*R, *x* ≠ −3

 (a) If f −1(*x*) = −5, find the value of *x*.

(b) Show that ff −1(*x*) = *x*

6. Functions f and g are such that

 f(*x*) = 3*x* + 2 g(*x*) = *x*2 + 1

 Find an expression for (fg)−1(*x*) [Hint: First find fg(*x*) ]

**MIXED EXERCISE:**

1. Here is a number machine.



(a) Work out the **output** when the input is 4

(b) Work out the **input** when the output is 11

(c) Show that there is a value of the input for which the input and the output have the

 same value.

2. Functions f and g is such that f(*x*) = 2*x* – 1 and g(*x*) = 

(a) Find the value of

(i) f(3)

(ii) fg(6)

(b) Express the inverse function in the form f −1(*x*)) = .....

(c) Express the composite function gf in the form gf(*x*) = .....

3. The function f is such that f(*x*) = 4*x* – 1

(a) Find f −1(*x*)

The function g is such that g(*x*) = *kx*2 where *k* is a constant.

(b) Given that fg(2) = 12,work out the value of *k*

4. Functions f and g are such that f(*x*) = 3(*x* – 4) and g(*x*) =  + 1

(a) Find the value of f(10)

(b)Find g–1(*x*)

(c)Show that ff(*x*) = 9*x* – 48

5. Given that f(*x*) = *x*2 and g(*x*) = *x* – 6, solve the equation fg(*x*) = g–1(*x*)

6. f and g are functions such that f(*x*) = 2*x* – 3 and g(*x*) = 1 + 

(a) Calculate f(−4)

(b) Given that f(*a*) = 5, find the value of *a*.

(c) Calculate gf(6).

(d) Find the inverse function g −1(*x*).

7. Functions f and g are such that

f(*x*) =  and g(*x*) = 

(a) Calculate fg(10)

(b) Find the inverse function g −1(*x*).

8. Functions f and g are such that

f(*x*) = 2*x* + 2 and g(*x*) = 2*x* – 5

(a) Find the composite function fg.

 Give your answer as simply as possible.

(b) Find the inverse function f −1(*x*).

(c) Hence, or otherwise, solve f −1(*x*).= g −1(*x*).

9. The function f is such that f(*x*) = 

(a) Find the value of f(2)

(b) Given that f(*a*) = , find the value of *a*.

The function g is such that g(*x*) = *x* + 2

(c) Find the function gf.

 Give your answer as a single algebraic fraction in its simplest form.

10. Functions f and g are such that f(*x*) = *x*² and g(*x*) = *x* – 3

(a) Find gf(*x*).

(b) Find the inverse function g −1(*x*).

(c) Solve the equation gf(*x*) = g −1(*x*).

11. The function f is such that f(*x*) = (*x* – 1)2

(a) Find f(8)

The function g is such that g(*x*) = 

(b) Solve the equation g(*x*) = 1.2

(c) (i) Express the inverse function g –1 in the form g –1(*x*) = .......

 (ii) Hence write down gg(*x*) in terms of *x*.

12. f is a function such that 

(a) Find 

g is a function such that g(*x*) = , *x* ≥ 1

(b) Find fg(*x*)

 Give your answer as simply as possible.

13. The function f is such that 

(a) Find f(8)

(b) Express the inverse function f –1 in the form f –1(*x*) = …

The function g is such that 

(c) Express the function gf in the form gf(*x*) = …

 Give your answer as simply as possible.

14.Functions f and g are such that f(*x*) = 3*x* – 2 and g(*x*) = 

(a) Express the inverse function f –1 in the form f –1(*x*) = ...

(b) Find gf(*x*)

 Simplify your answer.

15.Functions f and g are such that f(*x*) =  and g(*x*) = 

(a) Solve gf(*a*) = 3

(b) Express the inverse function g –1 in the form g –1(*x*) = ...

16. Functions g and h are such that g(*x*) =  and h(*x*) = *x* + 4

(a) Find the value of g(1)

(b) Find gh(*x*)

 Simplify your answer.

(c) Express the inverse function g –1 in the form g –1(*x*) = …

**ANSWERS**

**Exercise 1**

1. (a) 17 (b) 37 (c) −18

 (d) –8 (e) −0.5 (f) 3.5

2. (a) 12 (b) 32 (c) 0

 (d) 32 (e) −3.96 (f) −3.19

3. (a) 1 (b) −3 (c) −7

 (d) –1 (e) 1.256 (f) −5.375

4. (a)  (b)  (c) 3

 (d)  (e)  (f) 2

5. 3*x*2 + 10*x*

6. (a) f(− 1) = 8 and g(2) = 9 (b) *x* = − 6

7. (a) f(− 3) = 17 and g(− 5) = − 23 (b) *x* =  and *x* = 3

**Exercise 2**

1. (a) 4 − 2*x* (b) 8*x* + 17 (c)  (d) 2(*x* + 3)2

2. (a) 7 − 2*x* (b) 8*x* + 7 (c)  (d) 2*x*2 + 3

3. (a) – 1 (b) *a* = 3

4. (a)(i) (5 + *x*)2 (ii) 5 + *x*2 (b) *x* = –2

5. 3(*x*2 + 4) – 1 = (3*x* – 1)2 + 4

 3*x*2 + 12 – 1 = 9*x*2 – 6*x* + 1 + 4

 3*x*2 + 11 = 9*x*2 – 6*x* + 5

 6*x*2 – 6*x* – 6 = 0

 *x*2 – *x* – 1 = 0

6. *x* = 

7. (a)  (b) 

8. *x* = 1 and *x* = 8

**Exercise 3**

1. (a)  (b)  (c)  (d)  (e)  (f) 4 – *x* (g)  (h)  (i)  (j) 

2. (a)  (b) *x* = 0.5

3. (a)  (b) *x* = − 4 and *x* = 2

4. 

5. (a)  (b) 

6. 

**Mixed Exercise**

1. (a) 8 (b) 29 (c) 2

2. (a) (i) 5 (ii) 0 (b)  (c) 

3. (a)  (b) *k* = 

4. (a) 18 (b) 5(*x* – 1)

 (c) ff(*x*) = 3(3(*x* − 4) − 4) = 3(3*x* − 12− 4) = 3(3*x* − 16) = 9*x* − 48

5. *x* = 6 and *x* = 7

6. (a) – 11 (b) *a* = 4 (c) 4 (d) (*x* – 1)2

7. (a)  (b) *x*2 + 2

8. (a) 6*x* – 13 (b)  (c) *x* = −19

9. (a)  (b) *a* = 7 (c) 

10. (a) *x*2 – 3 (b) *x* + 3 (c) *x* = −2 and *x* = 3

11. (a) 49 (b) *x* = 6 (c) (i)  (ii)  *x*

12. (a) 0.8 (b) 

13. (a) 1 (b) 2*x* + 6 (c) 

14. (a)  (b) 

15. (a) *x* = 4 (b) 

16. (a)  (b)  (c) 