Core Maths C1

Revision Notes

November 2012

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1 Algebra

Indices

Rules of indices

$a^m \times a^n = a^{m+n}$ $\quad (a^m)^n = a^{mn}$ $\quad a^{1/n} = \sqrt[n]{a}$

$a^m \div a^n = a^{m-n}$ $\quad a^0 = 1$ $\quad a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

$a^{-n} = \dfrac{1}{a^n}$

Examples:

(i) $5^{-3} \times 5^4 = 5^{-3+4} = 5^1 = 5.$

$7^{-4} \times 7^{-2} = 7^{-4+-2} = 7^{-4-2} = 7^{-6} = \dfrac{1}{7^6}.$

(ii) $3^5 \div 3^{-2} = 3^{5--2} = 3^{5+2} = 3^7.$

$9^{-4} \div 9^6 = 9^{-4-6} = 9^{-10} = \dfrac{1}{9^{10}}$

$11^{-3} \div 11^{-5} = 11^{-3--5} = 11^{-3+5} = 11^2 = 121$

(iii) $(6^{-3})^4 = 6^{-3\times 4} = 6^{-12} = \dfrac{1}{6^{12}}.$

(iv) $64^{2/3} = \left(64^{1/3}\right)^2 = (4)^2 = 16$

(v) $125^{-2/3} = \dfrac{1}{125^{2/3}}$ since *minus* means *turn upside down*

$= \dfrac{1}{5^2},$ since 3 on bottom of fraction is cube root, $\sqrt[3]{125} = 5$

$= \dfrac{1}{25}$

Example: $(16^a) \div (8^b) = (2^4)^a \div (2^3)^b = 2^{4a} \div 2^{3b} = 2^{4a-3b}.$

Example: Find x if $9^{2x} = 27^{x+1}$.

Solution: First notice that $9 = 3^2$ and $27 = 3^3$ and so

$9^{2x} = 27^{x+1} \quad \Rightarrow \quad (3^2)^{2x} = (3^3)^{x+1}$

$\Rightarrow \quad 3^{4x} = 3^{3x+3}$

$\Rightarrow \quad 4x = 3x + 3 \quad \Rightarrow \quad x = 3.$

Surds

A surd is a 'nasty' root – i.e. a root which is not rational

Thus $\sqrt{64} = 8$, $\sqrt[3]{1/27} = 1/3$, $\sqrt[5]{-243} = -3$ are rational and **not surds**

and $\sqrt{5}$, $\sqrt[5]{45}$, $\sqrt[3]{-72}$ are irrational and **are surds**.

Simplifying surds

Example: To simplify $\sqrt{50}$ we notice that $50 = 25 \times 2 = 5^2 \times 2$

$$\Rightarrow \quad \sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}.$$

Example: To simplify $\sqrt[3]{40}$ we notice that $40 = 8 \times 5 = 2^3 \times 5$

$$\Rightarrow \quad \sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2 \times \sqrt[3]{5}.$$

Rationalising the denominator

Rationalising means getting rid of surds.

We remember that multiplying $(a + b)$ by $(a - b)$ gives $a^2 - b^2$ which has the effect of squaring **both** a and b at the same time!!

Example: Rationalise the denominator of $\dfrac{2+3\sqrt{5}}{3-\sqrt{5}}$.

Solution: $$\frac{2+3\sqrt{5}}{3-\sqrt{5}} = \frac{2+3\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{6+3\sqrt{5}\sqrt{5}+9\sqrt{5}+2\sqrt{5}}{3^2 - \sqrt{5}^2}$$

$$= \frac{21+11\sqrt{5}}{4}.$$

Quadratic functions

A quadratic function is a function $ax^2 + bx + c$, where a, b and c are constants and the highest power of x is 2.

Completing the square.

Method 1

Rule: 1] The coefficient of x^2 must be 1.

2] Halve the coefficient of x, square it then add it and subtract it.

Example: Complete the square in $x^2 - 6x + 7$.

Solution: 1] The coefficient of x^2 is already 1, 2] the coefficient of x is -6, halve it to give -3 then square to give 9 and add and subtract

$$x^2 - 6x + 7 = x^2 - 6x + 9 - 9 + 7$$
$$= (x-3)^2 - 2.$$

Notice that the minimum value of the expression is -2 when $x = 3$, since the minimum value of $(x-3)^2$ is 0.

Example: Complete the square in $3x^2 + 24x - 5$.

Solution: 1] The coefficient of x^2 is not 1, so we must 'fiddle' it to make it 1, and then go on to step 2].

$$3x^2 + 24x - 5 = 3(x^2 + 8x) - 5$$
$$= 3(x^2 + 8x + 4^2 - 4^2) - 5$$
$$= 3(x+4)^2 - 48 - 5$$
$$= 3(x+4)^2 - 53$$

Notice that the minimum value of the expression is -53 when $x = -4$, since the minimum value of $(x+4)^2$ is 0. Thus the vertex of the curve is at $(-4, -53)$.

Method 2

Example: Re-write $x^2 - 12x + 7$ in completed square form

Solution: We know that we need -6 (half the coefficient of x)

and that $(x-6)^2 = x^2 - 12x + 36$

$\Rightarrow$ $x^2 - 12x + 7 = (x-6)^2 - 27$

Notice that the minimum value of the expression is -27 when $x = 6$, since the minimum value of $(x-6)^2$ is 0. Thus the vertex of the curve is at $(6, -27)$.

Example: Express $3x^2 + 12x - 2$ in completed square form.

Solution: The coefficient of x^2 is not 1 so we must 'take 3 out'

$3x^2 + 12x - 2 = 3(x^2 + 4x) - 2$

and we know that $(x+2)^2 = x^2 + 4x + 4$, so $x^2 + 4x = (x+2)^2 - 4$

$\Rightarrow$ $3x^2 + 12x - 2 = 3(x^2 + 4x) - 2 = 3[(x+2)^2 - 4] - 2$

$=$ $3(x+2)^2 - 14$

Notice that the minimum value of the expression is -14 when $x = -2$, since the minimum value of $(x-2)^2$ is 0. Thus the vertex of the curve is at $(-2, -14)$.

Factorising quadratics

All the coefficients must be whole numbers (integers) then the factors will also have whole number coefficients.

Example: Factorise $10x^2 + 11x - 6$.

Solution: Looking at the $10x^2$ and the -6 we see that possible factors are

$(10x \pm 1)$,	$(10x \pm 2)$,	$(10x \pm 3)$,	$(10x \pm 6)$,
$(5x \pm 1)$,	$(5x \pm 2)$,	$(5x \pm 3)$,	$(5x \pm 6)$,
$(2x \pm 1)$,	$(2x \pm 2)$,	$(2x \pm 3)$,	$(2x \pm 6)$,
$(x \pm 1)$,	$(x \pm 2)$,	$(x \pm 3)$,	$(x \pm 6)$,

Also the -6 tells us that the factors must have opposite signs, and by trial and error or common sense

$$10x^2 + 11x - 6 = (2x + 3)(5x - 2).$$

Solving quadratic equations.

by factorising.

Example: $x^2 - 5x + 6 = 0 \Rightarrow (x-3)(x-2) = 0$

$\Rightarrow \quad x - 3 = 0$ or $x - 2 = 0 \Rightarrow x = 3$ or $x = 2$.

Example: $x^2 + 8x = 0 \Rightarrow x(x+8) = 0$

$\Rightarrow \quad x = 0$ or $x + 8 = 0 \Rightarrow x = 0$ or $x = {}^-8$

N.B. Do **not** divide through by x first: you will lose the root of $x = 0$.

by completing the square

Example: $x^2 - 6x - 3 = 0$

$\Rightarrow \quad x^2 - 6x = 3$

and $\quad (x-3)^2 = x^2 - 6x + 9$

$\Rightarrow \quad x^2 - 6x + 9 = 3 + 9 \Rightarrow (x-3)^2 = 12$

$\Rightarrow \quad (x-3) = \pm\sqrt{12} \Rightarrow x = 3 \pm \sqrt{12} = -0.464$ or 6.46.

by using the formula

always try to factorise first.

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: $3x^2 - x - 5 = 0$ will not factorise, so we use the formula with

$a = 3$, $b = -1$, $c = -5$

$$\Rightarrow \quad x = \frac{--1 \pm \sqrt{(-1)^2 - 4 \times 3 \times (-5)}}{2 \times 3} = -1.135 \text{ or } +1.468$$

The discriminant $b^2 - 4ac$

In the formula for the quadratic equation

$$ax^2 + bx + c = 0 \qquad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i) there will be two distinct real roots if $b^2 - 4ac > 0$

ii) there will be only one real root if $b^2 - 4ac = 0$

iii) there will be no real roots if $b^2 - 4ac < 0$

Example: For what values of k does the equation $3x^2 - kx + 5 = 0$ have

i) two distinct real roots,
ii) exactly one real root
iii) no real roots.

Solution: The discriminant $b^2 - 4ac = (-k)^2 - 4 \times 3 \times 5 = k^2 - 60$

$\Rightarrow$ i) for two distinct real roots $k^2 - 60 > 0$

$\Rightarrow \quad k^2 > 60 \qquad \Rightarrow \quad k < -\sqrt{60}$, or $k > +\sqrt{60}$

and ii) for only one real root $k^2 - 60 = 0$

$\Rightarrow \quad k = \pm\sqrt{60}$

and iii) for no real roots $k^2 - 60 < 0$

$\Rightarrow \quad -\sqrt{60} < k < +\sqrt{60}$.

Miscellaneous quadratic equations

a) $2\sin^2 x - \sin x - 1 = 0$.
Put $y = \sin x$ to give $2y^2 - y - 1 = 0$

$\Rightarrow (y - 1)(2y + 1) = 0 \quad \Rightarrow \quad y = 1$ or $y = -½$
$\Rightarrow \sin x = 1$ or $-½ \quad \Rightarrow \quad x = 90°, 210°$ or $330°$ from $0°$ to $360°$.

b) $3^{2x} - 10 \times 3^x + 9 = 0$

Notice that $3^{2x} = (3^x)^2$ and put $y = 3^x$ to give
$y^2 - 10y + 9 = 0 \qquad \Rightarrow \qquad (y - 9)(y - 1) = 0$
$\Rightarrow \quad y = 9$ or $y = 1$
$\Rightarrow \quad 3^x = 9$ or $3^x = 1 \qquad \Rightarrow \qquad x = 2$ or $x = 0$.

c) $y - 3\sqrt{y} + 2 = 0$. Put $\sqrt{y} = x$ to give

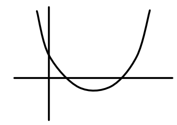
$x^2 - 3x + 2 = 0$ and solve to give $x = 2$ or 1

$\Rightarrow \qquad y = x^2 = 4$ or 1.

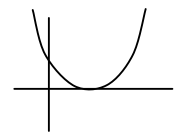
Quadratic graphs

a) If $a > 0$ the parabola will be 'the right way up' 

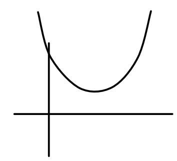
i) if $b^2 - 4ac > 0$ the curve will meet the x-axis in two points.



ii) if $b^2 - 4ac = 0$ the curve will meet the x-axis in only one point (it will *touch* the axis)

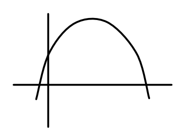


iii) if $b^2 - 4ac < 0$ the curve will not meet the x-axis.

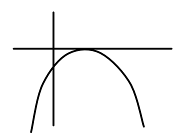


b) a) If $a < 0$ the parabola will be 'the wrong way up' 

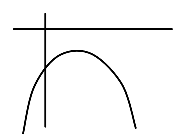
i) if $b^2 - 4ac > 0$ the curve will meet the x-axis in two points.



ii) if $b^2 - 4ac = 0$ the curve will meet the x-axis in only one point (it will *touch* the axis)



iii) if $b^2 - 4ac < 0$ the curve will not meet the x-axis.



Note: When sketching the curve of a quadratic function you should always show the value on the y-axis and

if you have factorised you should show the values where it meets the x-axis,

if you have completed the square you should give the co-ordinates of the vertex.

Simultaneous equations

Two linear equations

Example: Solve $3x - 2y = 4$ and $4x + 7y = 15$.

Solution: Make the coefficients of x (or y) equal then add or subtract the equations to eliminate x (or y).

Here 4 times the first equation gives $12x - 8y = 16$
and 3 times the second equation gives $\underline{12x + 21y = 45}$

Subtracting gives $-29y = -29 \quad \Rightarrow \quad y = 1$

Put $y = 1$ in equation one $\Rightarrow \quad 3x = 4 + 2y = 4 + 2 \quad \Rightarrow \quad x = 2$

Check in equation two: L.H.S. $= 4 \times 2 + 7 \times 1 = 15 =$ R.H.S.

One linear and one quadratic.

Find x (or y) from the Linear equation and substitute in the Quadratic equation.

Example: Solve $x - 2y = 3$, $x^2 - 2y^2 - 3y = 5$

Solution: From first equation $x = 2y + 3$

Substitute in second $\Rightarrow \quad (2y + 3)^2 - 2y^2 - 3y = 5$

$\Rightarrow \quad 4y^2 + 12y + 9 - 2y^2 - 3y = 5$

$\Rightarrow \quad 2y^2 + 9y + 4 = 0 \quad \Rightarrow \quad (2y + 1)(y + 4) = 0$

$\Rightarrow \quad y = -½$ or $y = -4$

$\Rightarrow \quad x = 2$ or $x = -5$ from the linear equation.

Check in quadratic for $x = 2$, $y = -½$

L.H.S. $= 2^2 - 2(-½)^2 - 3(-½) = 5 =$ R.H.S.

and for $x = -5$, $y = -4$

L.H.S. $= (-5)^2 - 2(-4)^2 - 3(-4) = 25 - 32 + 12 = 5 =$ R.H.S.

Inequalities

Linear inequalities

Solving algebraic inequalities is just like solving equations, add, subtract, multiply or divide the same number to, from, etc. BOTH SIDES

EXCEPT - if you multiply or divide both sides by a NEGATIVE number then you must TURN THE INEQUALITY SIGN ROUND.

Example: Solve $3 + 2x < 8 + 4x$

Solution: sub 3 from B.S. $\Rightarrow$ $2x < 5 + 4x$
sub $4x$ from B.S. $\Rightarrow$ $-2x < 5$
divide B.S. by -2 and **turn the inequality sign round**
$\Rightarrow$ $x > -2.5$.

Example: Solve $x^2 > 16$

Solution: We must be careful here since the square of a negative number is positive giving the full range of solutions as

$\Rightarrow$ $x < -4$ **or** $x > +4$.

Quadratic inequalities

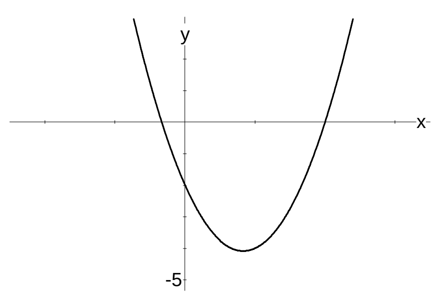
Always sketch a graph and find where the curve meets the x-axis

Example: Find the values of x which satisfy

$$3x^2 - 5x - 2 \geq 0.$$

Solution:

$3x^2 - 5x - 2 = 0$



$\Rightarrow$ $(3x + 1)(x - 2) = 0$

$\Rightarrow$ $x = -\frac{1}{3}$ or 2

We want the part of the curve which is above or on the x-axis

$\Rightarrow$ $x \leq -\frac{1}{3}$ **or** $x \geq 2$

3 Coordinate geometry

Distance between two points

Distance between P (a_1, b_1) and Q (a_2, b_2) is $\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

Gradient

Gradient of PQ is $m = \dfrac{b_2 - b_1}{a_2 - a_1}$

Equation of a line

Equation of the line PQ, above, is $y = mx + c$ and use a point to find c

or the equation of the line with gradient m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

or the equation of the line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$ (you do not need to know this one!!)

Parallel and perpendicular lines

Two lines are parallel if they have the same gradient
and they are perpendicular if the product of their gradients is -1.

Example: Find the equation of the line through (4½, 1) and perpendicular to the line joining the points A (3, 7) and B (6, -5).

Solution: Gradient of AB is $\dfrac{7 - -5}{3 - 6} = -4$

$\Rightarrow$ gradient of line perpendicular to AB is ¼ , (product of perpendicular gradients is -1)

so we want the line through (4½, 1) with gradient ¼ .

Using $y - y_1 = m(x - x_1)$ $\Rightarrow$ $y - 1 = \frac{1}{4}(x - 4\frac{1}{2})$

$\Rightarrow$ $4y - x = -\frac{1}{2}$ or $-2x + 8y + 1 = 0$.

4 Sequences and series

A *sequence* is any list of numbers.

Definition by a formula $x_n = f(n)$

Example: The definition $x_n = 3n^2 - 5$ gives

$x_{\mathbf{1}} = 3 \times \mathbf{1}^2 - 5 = -2, \quad x_{\mathbf{2}} = 3 \times \mathbf{2}^2 - 5 = 7, \quad x_{\mathbf{3}} = 3 \times \mathbf{3}^2 - 5 = 22, \; \dots .$

Definitions of the form $x_{n+1} = f(x_n)$

These have two parts:– (i) a starting value (or values)

(ii) a method of obtaining each term from the one(s) before.

Examples: (i) The definition $x_1 = 3$ and $x_n = 3x_{n-1} + 2$

defines the sequence 3, 11, 35, 107, . . .

(ii) The definition $x_1 = 1$, $x_2 = 1$, $x_n = x_{n-1} + x_{n-2}$

defines the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 65, ...

This is the Fibonacci sequence.

Series and Σ notation

A *series* is the sum of the first so many terms of a sequence.

For a sequence whose nth term is $x_n = 2n + 3$ the sum of the first n terms is a series

$$S_n = x_1 + x_2 + x_3 + x_4 \; \dots \; + x_n = 5 + 7 + 9 + 11 + \; \dots \; + (2n + 3)$$

This is written in Σ notation as $S_n = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (2i + 3)$ and is a *finite* series of n terms.

An *infinite* series has an infinite number of terms $S_\infty = \sum_{i=1}^{\infty} x_i$.

Arithmetic series

An *arithmetic series* is a series in which each term is a constant amount bigger (or smaller) than the previous term: this *constant amount* is called the *common difference*.

Examples: 3, 7, 11, 15, 19, 23, ... with common difference 4

28, 25, 22, 19, 16, 13, . . . with common difference –3.

Generally an arithmetic series can be written as

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + \; \dots$ upto n terms,

where the first term is a and the common difference is d.

The nth term $x_n = a + (n - 1)d$

The *sum* of the first n terms of the above arithmetic series is

$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$, or $S_n = \frac{n}{2}\left(a + L\right)$ where L is the **last** term.

Example: Find the nth term and the sum of the first 100 terms of the arithmetic series with 3rd term 5 and 7th term 17.

Solution: $x_7 = x_3 + 4 \times d$

$\Rightarrow$ $17 = 5 + 4 \times d$ $\Rightarrow$ $d = 3$

$\Rightarrow$ $x_1 = x_3 - 2 \times d$ $\Rightarrow$ $x_1 = 5 - 6 = -1$

$\Rightarrow$ nth term $x_n = a + (n-1)d = -1 + 3(n-1)$

and $\Rightarrow$ $S_{100} = {}^{100}/_2 \times (2 \times -1 + (100-1) \times 3) = 14750$.

Proof of the formula for the sum of an arithmetic series

You **must** know this proof.

First write down the general series and then write it down in reverse order

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-2)d) + (a+(n-1)d$$

$$\Rightarrow \quad S_n = (a+(n-1)d + (a+(n-2)d) + (a+(n-3)d) + \ldots + (a+d) + a \qquad \textbf{ADD}$$

$$\Rightarrow 2 \times S_n = (2a+(n-1)d) + (2a+(n-1)d) + (2a+(n-1)d) + \ldots (2a+(n-1)d) + (2a+(n-1)d)$$

$$\Rightarrow 2 \times S_n = n(2a+(n-1)d)$$

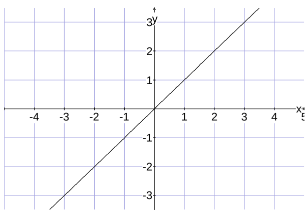
$$\Rightarrow \quad S_n = {}^{n}/_2 \times (2a+(n-1)d),$$

Which can be written as

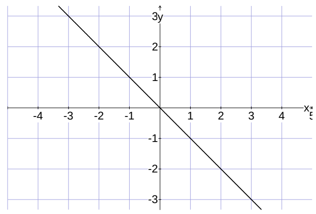
$$S_n = {}^{n}/_2 \times (a + a + (n-1)d) = {}^{n}/_2 \times (a + L), \text{ where } L \text{ is the last term.}$$

5 Curve sketching

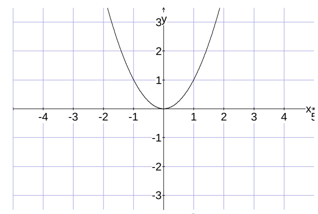
Standard graphs



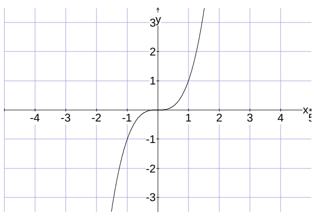
$y = x$



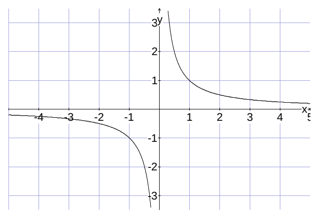
$y = -x$



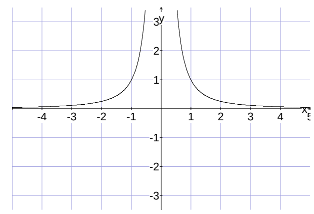
$y = x^2$



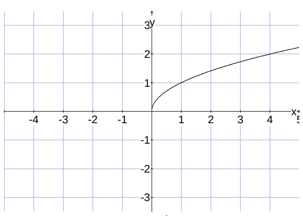
$y = x^3$



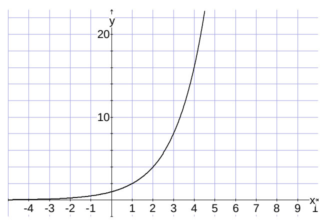
$y = 1/x$



$y = 1/x^2$

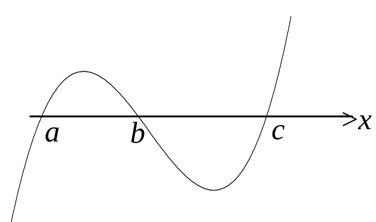


$y = \sqrt{x}$

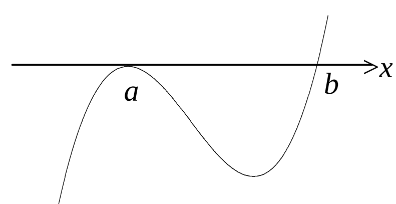


The exponential curve $y = 2^x$

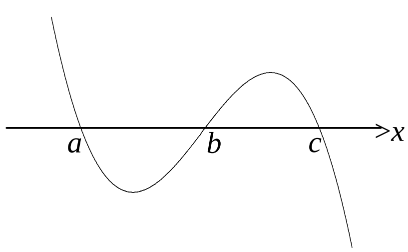
$y = 3x^2$ is like $y = x^2$ but steeper: similarly for $y = 5x^3$ and $y = {}^7/_x$, etc.



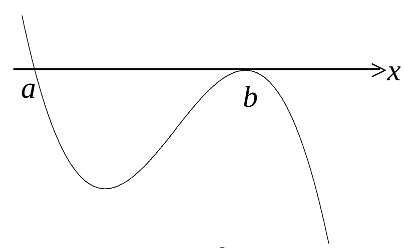
$y = (x - a)(x - b)(x - c)$



$y = (x - a)^2(x - b)$



$y = (a - x)(x - b)(x - c)$



$y = (a - x)(x - b)^2$

Transformations of graphs

Translations

(i) If the graph of $y = x^2 + 3x$ is translated through +5 in the y-direction the equation of the new graph is $y = x^2 + 3x + 5$;

and in general if we know that y is given by some formula involving x, which we write as $y = f(x)$, then the new curve after a translation through +5 units in the y direction is $y = f(x) + 5$.

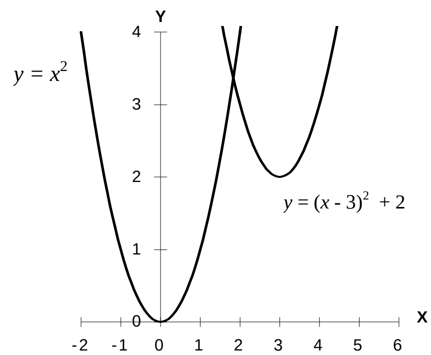
(ii) If the graph of $y = x^2 + 3x$ is translated through +5 in the x-direction the equation of the new graph is $y = (x - 5)^2 + 3(x - 5)$;

and if $y = f(x)$, then the new curve after a translation through +5 units in the x-direction is $y = f(x - 5)$:

i.e. we replace x by $(x - 5)$ everywhere in the formula for y:
note the minus sign, -5, which seems wrong but is correct!

Example:

The graph of $y = (x - 3)^2 + 2$ is the graph of $y = x^2$ after a translation of $\begin{pmatrix} +3 \\ +2 \end{pmatrix}$



In general $y = x^2$ or $y = f(x)$ becomes $y = (x - a)^2 + b$

or $y = f(x - a) + b$ after a translation through $\begin{pmatrix} a \\ b \end{pmatrix}$

Stretches

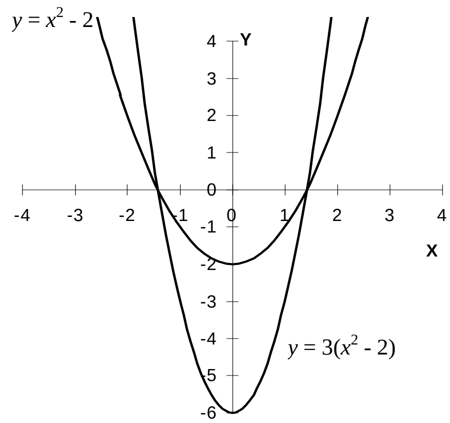
(i) If the graph of $y = x^2 + 3x$ is stretched by a factor of +5 in the y-direction.then the equation of the new graph is $y = 5(x^2 + 3x)$;

and in general if y is given by some formula involving x, which we write as $y = f(x)$,

then the new curve after a stretch by a factor of +5 in the y-direction is $y = 5 \times f(x)$.

Example:

The graph of $y = x^2 - 2$ becomes
$y = 3(x^2 - 2)$ after a stretch of factor 3 in the y-direction



In general $y = x^2$ or $y = f(x)$ becomes $y = ax^2$ or $y = af(x)$ after a stretch in the y-direction of factor a.

(ii) If the graph of $y = x^2 + 3x$ is stretched by a factor of +3 in the x-direction then the equation of the new graph is $y = \left(\frac{x}{3}\right)^2 + 3\left(\frac{x}{3}\right)$

and in general if y is given by some formula involving x, which we write as $y = f(x)$,

then the new curve after a a stretch by a factor of +3 in the x-direction is $y = f\left(\frac{x}{3}\right)$

i.e. we replace x by $\frac{x}{3}$ everywhere in the formula for y.

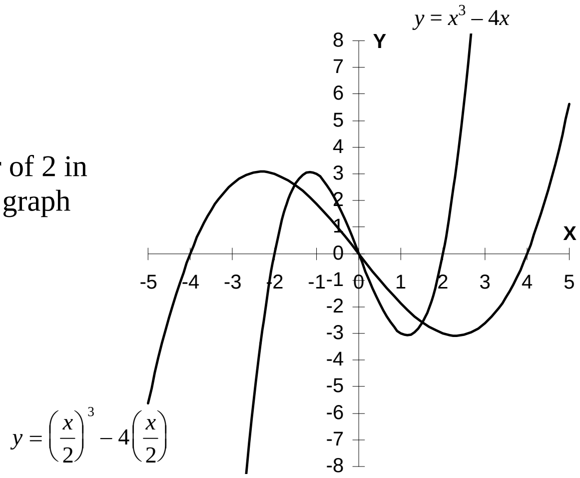
Example:

In this example the graph of

$y = x^3 - 4x$

has been stretched by a factor of 2 in the x-direction to form a new graph with equation

$$y = \left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)$$



(iii) Note that to stretch by a factor of ¼ in the x-direction we replace x by $\frac{x}{1/4} = 4x$ so that $y = f(x)$ becomes $y = f(4x)$

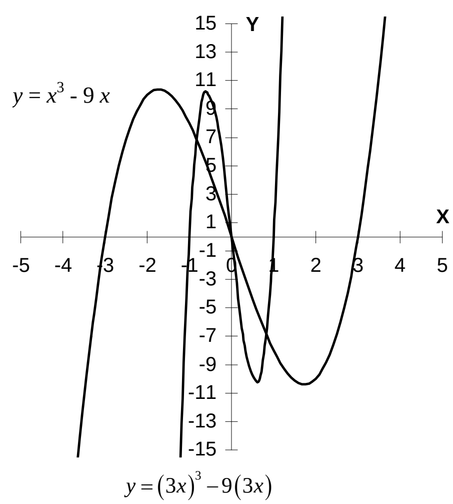
Example:

In this example the graph of $y = x^3 - 9x$

has been stretched by a factor of $\frac{1}{3}$ in the x-direction to form a new graph with equation

$y = (3x)^3 - 9(3x) = 27x^3 - 27x$,

The new equation is formed by replacing x by $\frac{x}{1/3} = 3x$ in the original equation.

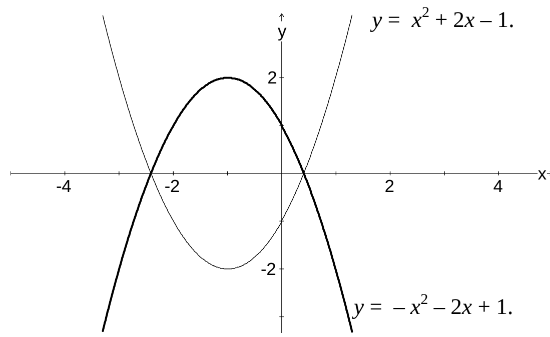


Reflections in the x-axis

When reflecting in the x-axis all the positive y-coordinates become negative and vice versa and so the image of $y = f(x)$ after reflection in the x-axis is $y = -f(x)$.

Example: The image of $y = x^2 + 2x - 1$ after reflection in the x-axis is

$$y = -f(x)$$
$$= -(x^2 + 2x - 1)$$
$$= -x^2 - 2x + 1$$



Reflections in the y–axis

When reflecting in the y-axis

the y-coordinate for $x = +3$ becomes the y-coordinate for $x = -3$ and the y-coordinate for $x = -2$ becomes the y-coordinate for $x = +2$.

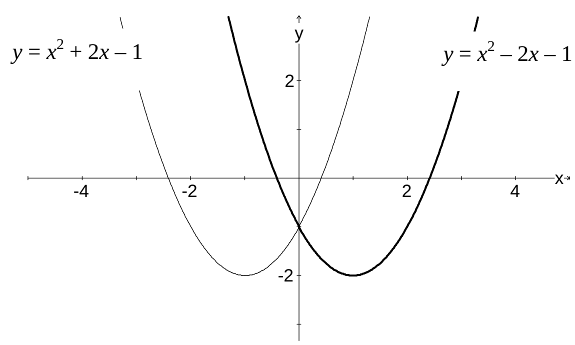
Thus the equation of the new graph is found by replacing x by $-x$ and the image of $y = f(x)$ after reflection in the y–axis is $y = f(-x)$.

Example: The image of

$y = x^2 + 2x - 1$

after reflection in the y-axis is

$$y = (-x)^2 + 2(-x) - 1$$
$$= x^2 - 2x - 1$$



Old equation	**Transformation**	**New equation**
$y = f(x)$	Translation through $\begin{pmatrix} a \\ b \end{pmatrix}$	$y = f(x - a) + b$
$y = f(x)$	Stretch with factor a in the y-direction.	$y = a \times f(x)$
$y = f(x)$	Stretch with factor a in the x-direction.	$y = f\left(\frac{x}{a}\right)$
$y = f(x)$	Stretch with factor $\frac{1}{a}$ in the x-direction.	$y = f(ax)$
$y = f(x)$	Reflection in the x-axis	$y = -f(x)$
$y = f(x)$	Reflection in the y-axis	$y = f(-x)$

6 Differentiation

General result

Differentiating is finding the gradient of the curve.

$$y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}, \qquad \text{or} \qquad f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$$

Examples:

a) $y = 3x^2 - 7x + 4$ $\qquad \frac{dy}{dx} = 6x - 7$

b) $f(x) = 7\sqrt{x} = 7x^{1/2}$ $\qquad f'(x) = 7 \times \frac{1}{2} x^{-1/2} = \frac{7}{2\sqrt{x}}$

c) $y = \frac{8}{x^3} = 8x^{-3}$ $\qquad \frac{dy}{dx} = 8 \times {}^-3x^{-4} = \frac{-24}{x^4}$

d) $f(x) = (2x + 1)(x - 3)$ $\qquad$ multiply out first

$= 2x^2 - 5x - 3$ $\qquad f'(x) = 4x - 5$

e) $y = \frac{3x^7 - 4x^2}{x^5}$ $\qquad$ split up first

$= \frac{3x^7}{x^5} - \frac{4x^2}{x^5} = 3x^2 - 4x^{-3}$ $\qquad \frac{dy}{dx} = 6x - -12x^{-4} = 6x + \frac{12}{x^4}$

Tangents and Normals

Tangents

Example:

Find the equation of the tangent to $y = 3x^2 - 7x + 5$ at the point where $x = 2$.

Solution:

We first find the gradient when $x = 2$.

$y = 3x^2 - 7x + 5 \quad \Rightarrow \quad \frac{dy}{dx} = 6x - 7$ and when $x = 2$, $\frac{dy}{dx} = 6 \times 2 - 7 = 5$.

so the gradient when $x = 2$ is 5. The equation is of the form $y = mx + c$ where m is the gradient so we have

$$y = 5x + c.$$

To find c we must find a point on the line, namely the point on the curve when $x = 2$.

When $x = 2$ the y – value on the curve is $3 \times 2^2 - 7 \times 2 + 5 = 3$,

i.e. when $x = 2$, $y = 3$.

Substituting these values in $y = 5x + c$ we get $3 = 5 \times 2 + c \quad \Rightarrow \quad c = -7$

$\Rightarrow$ the equation of the tangent is $y = 5x - 7$.

Normals

(The normal to a curve is the line at 90^o to the tangent at that point).

We first remember that if two lines with gradients m_1 and m_2 are perpendicular then $m_1 \times m_2 = -1$.

Example:

Find the equation of the normal to the curve $y = x + \frac{2}{x}$ at the point where $x = 2$.

Solution:

We first find the gradient of the tangent when $x = 2$.

$$y = x + 2x^{-1} \Rightarrow \frac{dy}{dx} = 1 - 2x^{-2} = 1 - \frac{2}{x^2}$$

$$\Rightarrow \text{ when } x = 2 \text{ gradient of the tangent is } m_1 = 1 - \frac{2}{4} = ½ .$$

If gradient of the normal is m_2 then $m_1 \times m_2 = -1 \Rightarrow ½ \times m_2 = -1 \quad \Rightarrow \quad m_2 = -2$

Thus the equation of the normal must be of the form $y = -2x + c$.

From the equation $y = x + \frac{2}{x}$ the value of y when $x = 2$ is $y = 2 + \frac{2}{2} = 3$

Substituting $x = 2$ and $y = 3$ in the equation of the normal $y = -2x + c$ we have

$3 = -2 \times 2 + c \quad \Rightarrow \quad c = 7$

$\Rightarrow$ equation of the normal is $y = -2x + 7$.

7 Integration

Indefinite integrals

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{provided that } n \neq -1$$

N.B. NEVER FORGET THE ARBITRARY CONSTANT + *C*.

Examples:

a) $\int \frac{4}{3x^6} \, dx = \int \frac{4}{3} x^{-6} \, dx$

$= \frac{4}{3} \times \frac{x^{-5}}{-5} + C = \frac{-4}{15x^5} + C$

b) $\int (3x-2)(x+1) \, dx = \int 3x^2 + x - 2 \, dx$ multiply out first

$= x^3 + ½\, x^2 - 2x + C.$

c) $\int \frac{x^9 + 5x^2}{x^5} \, dx = \int x^4 + 5x^{-3} \, dx$ split up first

$= \frac{x^5}{5} + 5 \times \frac{x^{-2}}{-2} + C = \frac{x^5}{5} - \frac{5}{2x^2} + C$

Finding the arbitrary constant

If you know the derivative (gradient) function and a point on the curve you can find *C*.

Example: Solve $\frac{dy}{dx} = 3x^2 - 5$, given that $y = 4$ when $x = 2$.

Solution: $y = \int 3x^2 - 5 \, dx = x^3 - 5x + C$ but $y = 4$ when $x = 2$

$\Rightarrow \quad 4 = 2^3 - 5 \times 2 + C \quad \Rightarrow \quad C = 6$

$\Rightarrow \quad y = x^3 - 5x + 6.$

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